1 Prove or Disprove

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in Note 2) you used.

(a) For all natural numbers n, if n is odd then n^2+3n is even.

(b) For all real numbers a, b, if $a+b \geq 20$ then $a \geq 17$ or $b \geq 3$.

(c) For all real numbers r, if r is irrational then $r+1$ is irrational.

(d) For all natural numbers n, $10n^3 > n!$.

(e) For all natural numbers a where a^5 is odd, then a is odd.

Solution:

1. True/False: For all natural numbers n, if n is odd then n^2+3n is even.

   True.
   Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. Substituting into the expression n^2+3n, we get $(2k+1)^2+3\times(2k+1)$. Simplifying the expression yields $4k^2+10k+4$. This can be rewritten as $2\times(2k^2+5k+2)$. Since $2k^2+5k+2$ is a natural number, by the definition of even numbers, n^2+3n is even. ■

2. True/False: For all real numbers a, b, if $a+b \geq 20$ then $a \geq 17$ or $b \geq 3$.

   True.
   Proof: We will use a proof by contraposition. Suppose that $a < 17$ and $b < 3$ (note that this is equivalent to $\neg(a \geq 17 \vee b \geq 3)$). Since $a < 17$ and $b < 3$, $a+b < 20$ (note that $a+b < 20$ is equivalent to $\neg(a+b \geq 20)$). Thus, if $a+b \geq 20$, then $a \geq 17$ or $b \geq 3$ (or both, as "or" is not "exclusive or" in this case). By contraposition, for all real numbers a, b, if $a+b \geq 20$ then $a \geq 17$ or $b \geq 3$. ■

3. True/False: For all real numbers r, if r is irrational then $r+1$ is irrational.

   True.
   Proof: We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form a/b where a and b are integers. Then r can be written as $(a-b)/b$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational. ■

4. True/False: For all natural numbers n, $10n^3 > n!$.

   False.
   Proof: We will use proof by counterexample. Let $n = 10$. $10 \times 10^3 = 10,000$. $(10!) = 3,628,800$. Since $10n^3 < n!$, the claim is false. ■

5. True/False: For all natural numbers a where a^5 is odd, then a is odd.

   True.
   Proof: This will be proof by contraposition. The contrapositive is "If a is even, then a^5 is even." Let a be even. By the definition of even, $a = 2k$. Then $a^5 = (2k)^5 = 2(16k^5)$, which implies a^5 even. By contraposition, for all natural numbers a where a^5 is odd, then a is odd. ■

2 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Solution:

(a) First we note that any integer can be written in one of the forms $3k$, $3k+1$, or $3k+2$. (Note that $3k+2$ is equal to $3(k+1)-1$. Since k is arbitary, we can treat these as equivalent forms).

   We can now prove the contrapositive: that any integer $m > 3$ of the form $3k$ must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5?

   For any prime $m > 5$, we can check if $m+2$ and $m-2$ are both prime. Note that if $m > 5$, then $m+2 > 3$ and $m-2 > 3$ so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

   Case 1: m is of the form $3k+1$. Then $m+2 = 3k+3$, which is divisible by 3. So $m+2$ is not prime.

   Case 2: m is of the form $3k-1$. Then $m-2 = 3k-3$, which is divisible by 3. So $m-2$ is not prime.

   So in either case, at least one of $m+2$ and $m-2$ is not prime.

3 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

(b) For all positive integers n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) For all positive natural numbers n, $\frac{5}{4} \cdot 8^n + 3^{3n-1}$ is divisible by 19.

Solution:

(a) The inequality is true for $n = 3$ because $8 > 7$. Let the inequality be true for $n = k$, such that $2^k > 2k+1$. Then,

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot (2k+1) = 4k+2$$

We know $2k > 1$ because k is a positive integer. Thus:

$$4k+2 = 2k+2k+2 > 2k+1+2 = 2k+3 = 2(k+1)+1$$

We've shown that $2^{k+1} > 2(k+1)+1$, which completes the inductive step.

(b) We can verify that the statement is true for $n = 1$. Assume the statement holds for $n = k$, so that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then we can write

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
&= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right) \\
&= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
&= (k+1)\left(\frac{(2k+3)(k+2)}{6}\right) \\
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6},
\end{aligned}$$

as desired. Since we've shown that the statement holds for $n = k+1$, our proof is complete.

(c) For $n = 1$, the statement is "$10+9$ is divisible by 19", which is true. Assume that the statement holds for $n = k$, such that $\frac{5}{4} \cdot 8^k + 3^{3k-1}$ is divisible by 19. Then,

$$\begin{aligned} \frac{5}{4} \cdot 8^{k+1} + 3^{3(k+1)-1} &= \frac{5}{4} \cdot 8 \cdot 8^k + 3^{3k+2} \\ &= 8 \cdot \frac{5}{4} \cdot 8^k + 3^3 \cdot 3^{3k-1} \\ &= 8 \cdot \frac{5}{4} \cdot 8^k + 8 \cdot 3^{3k-1} + 19 \cdot 3^{3k-1} \\ &= 8\left(\frac{5}{4} \cdot 8^k + 3^{3k-1}\right) + 19 \cdot 3^{3k-1} \end{aligned}$$

The first term is divisible by the inductive hypothesis, and the second term is clearly divisible by 19. This completes our proof, as we've shown the statement holds for $k+1$.

4 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

(c) Why does the hypothesis in part (b) imply the conclusion from part (a)?

Solution:

(a) Let's try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.
Inductive Step: For some $n \geq 1$, we assume $a_n \leq 3^{2^n}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.
Inductive Step: For some $n \geq 1$ we assume $a_n \leq 3^{2^n - 1}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n-1})^2 = 3 \times 3^{2\times(2^n-1)} = 3 \times 3^{2^{n+1}-2} = 3^{2^{n+1}-1}.$$

This is exactly the induction hypothesis for $n+1$.

(c) For every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n-1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved in part (b) does indeed imply what we wanted to prove in part (a).