1 Set Operations

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: \( \{a/b : a, b \in \mathbb{Z} \land b \neq 0\} \)
- $\mathbb{Z}$, the set of integers: \{\ldots, -2, -1, 0, 1, 2, \ldots\}
- $\mathbb{N}$, the set of natural numbers: \{0, 1, 2, 3, \ldots\}

(a) Given a set $A = \{1, 2, 3, 4\}$, what is $\mathcal{P}(A)$ (Power Set)?

(b) Given a generic set $B$, how do you describe $\mathcal{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathcal{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What is $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

Solution:

(a) $\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$,
\[\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$

(b) $\mathcal{P}(B) = \{T \mid T \subseteq B\}$

(c) \{\} or $\emptyset$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

(f) The set of irrational numbers

(g) $\emptyset$
Note 0

2. Preserving Set Operations

For a function $f$, define the image of a set $X$ to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set $Y$ to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which $A$ and $B$ are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets $X$ and $Y$, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x) ((x \in X) \implies (x \in Y)).$

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that $A$ is a subset of $B$, $A \subseteq B$, and that $B$ is a subset of $A$, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose $x$ is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both $A$ and $B$, so $x$ lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, $x$ is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(b) Suppose $x$ is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(c) Suppose $x \in A \cap B$. Then, $x$ lies in both $A$ and $B$, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but $A$ and $B$ are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(d) Suppose $y \in f(A) \setminus f(B)$. Since $y$ is not in $f(B)$, there are no elements in $B$ which map to $y$. Let $x$ be any element of $A$ that maps to $y$; by the previous sentence, $x$ cannot lie in $B$. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A \setminus B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A \setminus B) = \{0\}$. Hence, $f(A) = f(B) = \{0\}$, so $f(A \setminus B) = \emptyset$. 

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3 Inverses and Bijections

Recall that a function \( f : A \rightarrow B \) is a bijection if it is an injection and a surjection, and it is invertible if there is a function \( g : B \rightarrow A \) so that \( g \circ f = \text{id}_A \) and \( f \circ g = \text{id}_B \), where \( \text{id}_A : A \rightarrow A \) and \( \text{id}_B : B \rightarrow B \) are the identity functions.

(a) Prove that if \( f : A \rightarrow B \) is invertible then it is a bijection.

(b) Prove that if \( f : A \rightarrow B \) is a bijection then it is invertible.

(c) Let \( g : B \rightarrow A \) be the inverse function for some bijection \( f \). Is \( g \) necessarily a bijection?

Solution:

(a) Suppose \( g : B \rightarrow A \) is the inverse of \( f \). First, we show \( f \) is injective. Suppose \( f(x) = f(y) \) for some \( x,y \in A \). Then, \( g(f(x)) = g(f(y)) \). Since \( g \) is the inverse of \( f \), \( g \circ f = \text{id}_A \), so we get \( x = y \). Thus, \( f \) is injective. Next, we show \( f \) is surjective. Consider any \( b \in B \). Then, \( g(b) \in A \) is such that \( f(g(b)) = b \) because \( f \circ g = \text{id}_B \). So, \( f \) is surjective. Thus, \( f \) is a bijection.

(b) Since \( f \) is surjective, every element \( b \in B \) is mapped to by something (in other words, the preimage \( f^{-1}({\{b\}}) \) is nonempty). Since \( f \) is injective, every element \( b \in B \) is mapped to by at most one thing (in other words, the preimage \( f^{-1}({\{b\}}) \) has cardinality at most 1). Combining these facts, for each \( b \in B \), \( f^{-1}({\{b\}}) = \{a\} \) for some \( a \in A \). Define \( g : B \rightarrow A \) so that \( g(b) \) is the unique element in \( f^{-1}({\{b\}}) \) for each \( b \in B \). We claim \( g \) is the inverse of \( f \).

First, consider \( g \circ f : A \rightarrow B \). For any \( a \in A \), \( g(f(a)) \) is the unique element in \( f^{-1}({\{f(a)\}}) \) which must be \( a \) since \( f \) maps \( a \) to \( f(a) \), so \( g(f(a)) = a \) and thus \( g \circ f = \text{id}_A \). Now, consider \( f \circ g : B \rightarrow A \). For any \( b \in B \), \( g(b) \) is the unique element in \( f^{-1}({\{b\}}) \), which means \( f \) maps it to \( b \). Thus, \( f(g(b)) = b \) and so \( f \circ g = \text{id}_B \).

(c) Yes! The condition for being an inverse is symmetric, so \( f \) is the inverse of \( g \). Therefore, \( g \) is invertible and hence a bijection by part (a).

4 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

Solution: We prove the statement by contradiction. Suppose that \( ab = c \), where \( a \neq 0 \) is rational, \( b \) is irrational, and \( c \) is rational. Since \( a \) and \( b \) are not zero (because 0 is rational), \( c \) is also non-zero. Thus, we can express \( a = \frac{p}{q} \) and \( c = \frac{r}{s} \), where \( p,q,r \), and \( s \) are non-zero integers. Then

\[
b = \frac{c}{a} = \frac{rq}{ps},
\]

which is the ratio of two nonzero integers, giving that \( b \) is rational. This contradicts our initial assumption, so we conclude that the product of a non-zero rational number and an irrational number is irrational.