Graph Theory II

Note 5

Planar Graph: A graph which can be drawn on a plane with no crossings. Planar graphs have faces, which are regions of the plane where any two points can be connected by a path without crossing the drawing of an edge. Note that it is fine if there is another drawing of a planar graph with crossings; as long as there exists a drawing of a graph without crossing, then the graph is planar.

A planar graph G with v vertices and e edges with a planar drawing with f faces satisfy the following:

- Euler's formula: $v+f=e+2$
- $\sum_{i=1}^{f} s_i = 2e$ where s_i is the number of edges (sides) bordering face i. (This is somewhat like the degree-sum formula in that each edge is a side to two faces.)
- If planar, then $e \leq 3v-6$
- If bipartite planar, then $e \leq 2v-4$
- Graphs are non-planar iff they contain K_5 or $K_{3,3}$ (the complete graph on 5 vertices or the complete bipartite graph on 3 vertices in each set) as a subgraph
- All planar graphs can be vertex colored in at most 4 colors

Complete graph: The complete graph on n vertices, denoted by K_n, contains an edge between every pair of vertices.

Bipartite graph: A graph G with two sets of vertices such that each edge is incident to one vertex from each set.

Tree: A graph is a tree iff it satisfies any of the following:

- connected and acyclic
- connected and has $|V|-1$ edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

Hypercube: The hypercube of dimension n has 2^n vertices, each labeled by a length n bitstring. Edges connect vertices that differ by exactly one bit. A hypercube of dimension $n+1$ can be recursively constructed by creating two copies of an n-dimensional hypercube and connecting corresponding vertices by an edge.

1 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.

(b) G requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

(d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in G has degree at most 2.

2 Short Answers

Note 5

In each part below, provide the number/equation and brief justification.

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from K_6 to get a tree?

(c) The Euler's formula $v - e + f = 2$ requires the planar graph to be connected. What is the analogous formula for planar graphs wth k connected components?

3 Graph Coloring

Note 5

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

4 Hypercubes

Note 5

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.