1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1 + x)^n \geq 1 + nx$.

**Solution:**

- **Base Case:** When $n = 0$, the claim holds since $(1 + x)^0 \geq 1 + 0x$.
- **Inductive Hypothesis:** Assume that $(1 + x)^k \geq 1 + kx$ for some value of $n = k$ where $k \in \mathbb{N}$.
- **Inductive Step:** For $n = k + 1$, we can show the following:

  $$(1 + x)^{k+1} = (1 + x)^k(1 + x) \geq (1 + kx)(1 + x)$$

  $$\geq 1 + kx + x + kx^2$$

  $$\geq 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$$

  By induction, we have shown that $\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$.

2 Strengthen Induction

Show by induction that $\sum_{i=1}^{n} \frac{1}{i^3} \leq 2$.

**Solution:**

We will prove the stronger theorem that $\sum_{i=1}^{n} \frac{1}{i^3} \leq 2 - \frac{1}{n^2}$ by induction.

Base case: For $n = 1$, $1 \leq 2 - \frac{1}{1^2} = 1$.

Induction hypothesis: $\sum_{i=1}^{k} \frac{1}{i^3} \leq 2 - \frac{1}{k^2}$. $\sum_{i=1}^{k+1} \frac{1}{i^3} \leq (2 - \frac{1}{k^2}) + \frac{1}{(k+1)^3} = 2 - (\frac{1}{k^2} - \frac{1}{(k+1)^2})$ Now to complete the proof, we need to prove

$$1 \leq k^2 - \frac{1}{(k+1)^3}$$

$$1 \leq \frac{1}{k^2} - \frac{1}{(k+1)^3}$$

$$k + 2 \leq \frac{1}{k^2}$$

$$k^3 + 2k^2 \leq \frac{(k + 1)^3}{(k^2)(k+1)^3}$$

$$0 \leq \frac{k^2 + 3k + 1}{(k^2)(k+1)^3}$$

(1)
Thus the inequality holds, and the statement follows.

3 Binary Numbers

Prove that every positive integer \( n \) can be written in binary. In other words, prove that we can write

\[ n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0, \]

where \( k \in \mathbb{N} \) and \( c_i \in \{0, 1\} \) for all \( i \leq k \).

Solution:

Prove by strong induction on \( n \).

The key insight here is that if \( n \) is divisible by 2, then it is easy to get a bit string representation of \( (n + 1) \) from that of \( n \). However, if \( n \) is not divisible by 2, then \( (n + 1) \) will be, and its binary representation will be more easily derived from that of \( (n + 1)/2 \). More formally:

• Base Case: \( n = 1 \) can be written as \( 1 \times 2^0 \).

• Inductive Step: Assume that the statement is true for all \( 1 \leq m \leq n \), where \( n \) is arbitrary. Now, we need to consider \( n + 1 \). If \( n + 1 \) is divisible by 2, then we can apply our inductive hypothesis to \( (n + 1)/2 \) and use its representation to express \( n + 1 \) in the desired form.

\[
(n + 1)/2 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0
\]

\[
n + 1 = 2 \cdot (n + 1)/2 = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0.
\]

Otherwise, \( n \) must be divisible by 2 and thus have \( c_0 = 0 \). We can obtain the representation of \( n + 1 \) from \( n \) as follows:

\[
n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 0 \cdot 2^0
\]

\[
n + 1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 1 \cdot 2^0
\]

Therefore, the statement is true.

Here is another alternate solution emulating the algorithm of converting a decimal number to a binary number.

• Base Case: \( n = 1 \) can be written as \( 1 \times 2^0 \).

• Inductive Step: Assume that the statement is true for all \( 1 \leq m \leq n \), for arbitrary \( n \). We show that the statement holds for \( n + 1 \). Let \( 2^m \) be the largest power of 2 such that \( n + 1 \geq 2^m \). Thus, \( n + 1 < 2^{m+1} \). We examine the number \( (n + 1) - 2^m \). Since \( (n + 1) - 2^m < n + 1 \), the inductive hypothesis holds, so we have a binary representation for \( (n + 1) - 2^m \). Also, since \( n + 1 < 2^{m+1}, \ (n + 1) - 2^m < 2^m \), so the largest power of 2 in the representation of \( (n + 1) - 2^m \) is \( 2^{m-1} \). Thus, by the inductive hypothesis,

\[
(n + 1) - 2^m = c_{m-1} \cdot 2^{m-1} + c_{m-2} \cdot 2^{m-2} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,
\]
and adding $2^m$ to both sides gives

$$n + 1 = 2^m + c_{m-1} \cdot 2^{m-1} + c_{m-2} \cdot 2^{m-2} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

which is a binary representation for $n + 1$. Thus, the induction is complete.

Another intuition is that if $x$ has a binary representation, $2x$ and $2x + 1$ do as well: shift the bits and possibly place 1 in the last bit. The above induction could then have proceeded from $n$ and used the binary representation of $\lfloor n/2 \rfloor$, shifting and possibly setting the first bit depending on whether $n$ is odd or even.

Note: In proofs using simple induction, we only use $P(n)$ in order to prove $P(n + 1)$. Simple induction gets stuck here because in order to prove $P(n + 1)$ in the inductive step, we need to assume more than just $P(n)$. This is because it is not immediately clear how to get a representation for $P(n + 1)$ using just $P(n)$, particularly in the case that $n + 1$ is divisible by 2. As a result, we assume the statement to be true for all of 1, 2, \ldots, $n$ in order to prove it for $P(n + 1)$. 