1 Degree Sequences

The degree sequence of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is (3, 2, 2, 2, 1).

For each of the parts below, determine if there exists a simple undirected graph $G$ (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

(a) (3, 3, 2, 2)
(b) (3, 2, 2, 2, 1, 1)
(c) (6, 2, 2, 2)
(d) (4, 4, 3, 2, 1)

2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say $L$ and $R$), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{\text{green vertices}\}$ and $R = \{\text{red vertices}\}$), and a non-bipartite graph.

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph $G$ being a bipartite implies that $G$ has no tours of odd length).
3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of $G$ that have odd degree is even. Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in $G$). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

![Figure 1: A bipartite graph (left) and a non-bipartite graph (right).](image)
(iii) Induction on $n = |V|$ (number of vertices)