CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 2B

1 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

(b) A simple graph obtained by adding an edge between two vertices of a tree creates a cycle.

(c) Adding an edge in a connected graph creates exactly one new cycle.

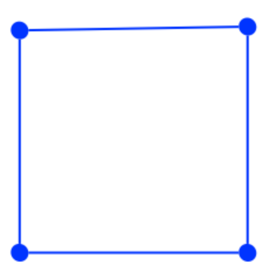
Solution:

(a) **True.**
Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

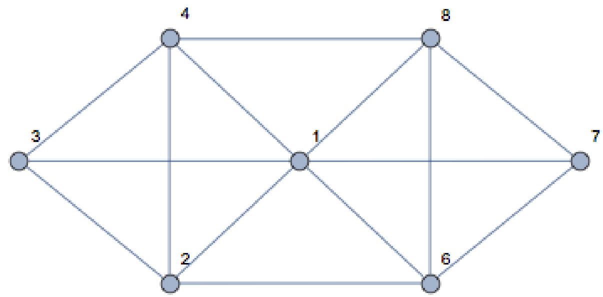
(b) **True.**
Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

Adding an edge between a pair of vertices x, y connected by an edge creates a graph that is not simple. So, we do not need to consider this case. To be sure, the condition is not really needed as the existence of two edges beteen x and y also creates a cycle.

(c) **False.**
In the following graph adding an edge creates two cycles.



2 Eulerian Tour and Eulerian Walk



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

Solution:

(a) No. Two vertices have odd degree.

(b) Yes. One of the two vertices with odd degree must be the starting vertex, and the other one must be the ending vertex. For example: $3,4,2,1,3,2,6,1,4,8,1,7,8,6,7$ will be an Eulerian walk (the numbers are the vertices visited in order). Note that there are 14 edges in the graph.

(c) This solution is long and in depth. Please read slowly, and don't worry if it takes multiple read-throughs since this is dense mathematical text.

An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and has at most two odd degree vertices. Note that there is no graph with only one odd degree vertex (this is a result of the Handshake lemma, which we will prove in the next question). An Eulerian tour is also an Eulerian walk which starts and ends at the same vertex. We have already seen in the lectures, that an undirected graph G has an Eulerian tour if and only if G is connected (except for isolated vertices) and all its vertices have even degree. We will now prove that a graph G has an Eulerian walk with distinct starting and ending vertex, if and only if it is connected (except for isolated vertices) and has exactly two odd degree vertices.

Justifications: *Only if.* Suppose there exists an Eulerian walk, say starting at u and ending at v (note that u and v are distinct). Then all the vertices that lie on this walk are connected to each other and all the vertices that do not lie on this walk (if any) must be isolated. Thus the graph is connected (except for isolated vertices). Moreover, every intermediate visit to a vertex in this walk is being paired with two edges, and therefore, except for u and v, all other vertices must be of even degree.

If. First, note that for a connected graph with no odd degree vertices, we have shown in the lectures that there is an Eulerian tour, which implies an Eulerian walk. Thus, let us consider the case of two odd degree vertices.

Solution 1: Take the two odd degree vertices u and v, and add a vertex w with two edges (u,w) and (w,v). The resulting graph G' has only vertices of even degree (we added one to the degree of u and v and introduced a vertex of degree 2) and is still connected. So, we can find an Eulerian tour on G'. Now, delete the component of the tour that uses edges (u,w) and (w,v). The part of the tour that is left is now an Eulerian walk from u to v on the original graph, since it traverses every edge on the original graph.

Solution 2: Alternatively, we can construct an algorithm quite similar to the FindTour algorithm with splicing described in the graphs note.

Suppose G is connected (except for isolated vertices) and has exactly two odd degree vertices, say u and v. First remove the isolated vertices if any. Since u and v belong to a connected component, one can find a path from u to v. Consider the graph obtained by removing the edges of the path from the graph. In the resulting graph, all the vertices have even degree. Hence, for each connected component of the residual graph, we find an Eulerian tour. (Note that the graph obtained by removing the edges of the path can be disconnected.) Observe that an Eulerian walk is simply an edge-disjoint walk that covers all the edges. What we just did is decomposing all the edges into a path from u to v and a bunch of edge-disjoint Eulerian tours. A path is clearly an edge-disjoint walk. Then, given an edge-disjoint walk and an edge-disjoint tour such that they share at least one common vertex, one can combine them into an edge-disjoint walk simply by augmenting the walk with the tour at the common vertex. Therefore we can combine all the edge-disjoint Eulerian tours into the path from u to v to make up an Eulerian walk from u to v.

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V,E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

Solution:

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v\in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\mathsf{c}}$, so we can write

$$\sum_{v\in V_{\text{odd}}(G)} d_v = 2m - \sum_{v\notin V_{\text{odd}}(G)} d_v.$$

Both terms in the right-hand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the left-hand side $\sum_{v\in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u,v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u,v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u,v\} & \text{if } u,v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u,v\} & \text{if } u,v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\mathsf{c}}$ had even degree in

G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (ii), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

4 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Solution:

Proof using induction on the number of vertices n.

Base case $n = 2$. A tree with two vertices has only one edge and is a bipartite graph by partitioning the two vertices into two separate parts.

Inductive hypothesis. Assume that all trees with k vertices for an arbitrary $k \geq 2$ is bipartite.

Inductive step. Consider a tree $T = (V, E)$ with $k+1$ vertices. We know that every tree must have at least two leaves, so remove one leaf u and the edge connected to u, say edge e. The resulting graph $T - u$ is a tree with k vertices and is bipartite by the inductive hypothesis. Thus there exists a partitioning of the vertices $V = R \cup L$ such that there does not exist an edge that connects two vertices in L or two vertices in R. Now when we add u back to the graph. If edge e connects u with a vertex in L then let $L' = L$ and $R' = R \cup \{u\}$. On the other hand if edge e connects u with a vertex in R then let $L' = L \cup \{u\}$ and $R' = R$. L' and R' gives us the required partition to show that T is bipartite. This completes the inductive step and hence by induction we get that all trees with at least 2 vertices are bipartite.