CS 70 Discrete Mathematics and Probability Theory
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1 GCD Proof

Let n, x be positive integers. Prove that x has a multiplicative inverse modulo n if and only if $\gcd(n,x) = 1$. (Hint: Remember an iff needs to be proven both directions. The gcd cannot be 0 or negative.)

Solution: If x has a multiplicative inverse modulo n, then $\gcd(n,x) = 1$.

Given that x has a multiplicative inverse modulo n, we can proceed as follows:

Assume for the sake of contradiction that the gcd, d, is greater than 1.

$$\begin{aligned} xa &\equiv 1 \pmod{n} \\ xa &= bn+1 \\ \frac{xa}{d} &= \frac{bn+1}{d} \\ \frac{xa}{d} &= \frac{bn}{d} + \frac{1}{d} \end{aligned}$$

We've reached a contradiction because xa/d and bn/d must both be integers, however, $1/d$ is not. Therefore we've reached a contradiction, and because the gcd cannot be 0 or negative, it must be 1.

If $\gcd(n,x) = 1$, then x has a multiplicative inverse modulo n. One proof using bijections is already shown in lecture. Here's another one based on egcd:

We know that when we run egcd on n and x, the output is integers $a, b \in \mathbb{Z}$ such that

$$\begin{aligned} an + bx &= 1, \\ bx &\equiv 1 \pmod{n}. \end{aligned}$$

Thus, x has a multiplicative inverse b.

2 Extended Euclid

In this problem we will consider the extended Euclid's algorithm. The bolded numbers below keep track of which numbers appeared as inputs to the gcd call. Remember that we are interested in writing the GCD as a linear combination of the original inputs, so we don't want to accidentally simplify the expressions and eliminate the inputs.

(a) Note that $x \bmod y$, by definition, is always x minus a multiple of y. So, in the execution of Euclid's algorithm, each newly introduced value can always be expressed as a "combination" of the previous two, like so:

$$\begin{aligned} \gcd(54,17) &= \gcd(17,3) & \mathbf{3} &= 1\times\mathbf{54} - 3\times\mathbf{17} \\ &= \gcd(3,2) & \mathbf{2} &= 1\times\mathbf{17} - \underline{\qquad}\times\mathbf{3} \\ &= \gcd(2,1) & \mathbf{1} &= 1\times\mathbf{3} - \underline{\qquad}\times\mathbf{2} \\ &= \gcd(1,0) & [\mathbf{0} &= 1\times\mathbf{2} - \underline{\qquad}\times\mathbf{1}] \\ &= 1. \end{aligned}$$

(Fill in the blanks)

(b) Recall that our goal is to fill out the blanks in

$$1 = \underline{\qquad}\times\mathbf{54} + \underline{\qquad}\times\mathbf{17}.$$

To do so, we work back up from the bottom, and express the gcd above as a combination of the two arguments on each of the previous lines:

$$\begin{aligned} 1 &= \underline{\qquad}\times\mathbf{3} + \underline{\qquad}\times\mathbf{2} \\ &= \\ &= \underline{\qquad}\times\mathbf{17} + \underline{\qquad}\times\mathbf{3} \\ &= \\ &= \underline{\qquad}\times\mathbf{54} + \underline{\qquad}\times\mathbf{17} \end{aligned}$$

(c) In the same way as just illustrated in the previous two parts, calculate the gcd of 17 and 39, and determine how to express this as a "combination" of 17 and 39.

(d) What does this imply, in this case, about the multiplicative inverse of 17, in arithmetic mod 39?

Solution:

(a) Filling in the blanks,

$$\begin{aligned} \mathbf{3} &= 1\times\mathbf{54} - 3\times\mathbf{17} \\ \mathbf{2} &= 1\times\mathbf{17} - 5\times\mathbf{3} \\ \mathbf{1} &= 1\times\mathbf{3} - 1\times\mathbf{2} \\ [\mathbf{0} &= 1\times\mathbf{2} - 2\times\mathbf{1}] \end{aligned}$$

It may be easier to think about this in a rearranged form: $\mathbf{54} = 3\times\mathbf{17} + \mathbf{3}$, etc.; this directly corresponds to the $54 \bmod 17 = 3$ operation in the forward pass, and the desired blank comes from $\lfloor 54/17 \rfloor$.

(b) Working our way backward up the equalities and substituting them in, we have

$$\begin{aligned}1 &= 1\times\mathbf{3}-1\times\mathbf{2}\\ &= 1\times\mathbf{3}-1\times(1\times\mathbf{17}-5\times\mathbf{3})\\ &= -1\times\mathbf{17}+6\times\mathbf{3}\\ &= -1\times\mathbf{17}+6\times(1\times\mathbf{54}-3\times\mathbf{17})\\ &= 6\times\mathbf{54}-19\times\mathbf{17}\end{aligned}$$

(c) Doing the forward pass,

$$\begin{aligned}\gcd(39,17) &= \gcd(17,5) & \mathbf{5} &= 1\times\mathbf{39}-2\times\mathbf{17}\\ &= \gcd(5,2) & \mathbf{2} &= 1\times\mathbf{17}-3\times\mathbf{5}\\ &= \gcd(2,1) & \mathbf{1} &= 1\times\mathbf{5}-2\times\mathbf{2}\\ &= \gcd(1,0) & [\mathbf{0} &= 1\times\mathbf{2}-2\times\mathbf{1}]\end{aligned}$$

Going back up, we have

$$\begin{aligned}\mathbf{1} &= 1\times\mathbf{5}-2\times\mathbf{2}\\ &= 1\times\mathbf{5}-2\times(1\times\mathbf{17}-3\times\mathbf{5})\\ &= -2\times\mathbf{17}+7\times\mathbf{5}\\ &= -2\times\mathbf{17}+7\times(1\times\mathbf{39}-2\times\mathbf{17})\\ &= 7\times\mathbf{39}-16\times\mathbf{17}\end{aligned}$$

This leaves us with a final answer of $1 = 7\times\mathbf{39}-16\times\mathbf{17}$.

(d) It is equal to $-16 \bmod 39$, which is equal to $23 \bmod 39$.

3 Chinese Remainder Theorem Practice

In this question, you will solve for a natural number x such that,

$$\begin{aligned}x &\equiv 2 \pmod 3\\ x &\equiv 3 \pmod 5\\ x &\equiv 4 \pmod 7\end{aligned} \tag{1}$$

(a) Suppose you find 3 natural numbers a,b,c that satisfy the following properties:

$$a \equiv 2 \pmod 3\ ;\ a \equiv 0 \pmod 5\ ;\ a \equiv 0 \pmod 7, \tag{2}$$
$$b \equiv 0 \pmod 3\ ;\ b \equiv 3 \pmod 5\ ;\ b \equiv 0 \pmod 7, \tag{3}$$
$$c \equiv 0 \pmod 3\ ;\ c \equiv 0 \pmod 5\ ;\ c \equiv 4 \pmod 7. \tag{4}$$

Show how you can use the knowledge of a, b and c to compute an x that satisfies (1).

In the following parts, you will compute natural numbers a,b and c that satisfy the above 3 conditions and use them to find an x that indeed satisfies (1).

(b) Find a natural number a that satisfies (2). In particular, an a such that $a \equiv 2 \pmod{3}$ and is a multiple of 5 and 7. It may help to approach the following problem first:

   (b.i) Find a^*, the multiplicative inverse of 5×7 modulo 3. What do you see when you compute $(5 \times 7) \times a^*$ modulo 3, 5 and 7? What can you then say about $(5 \times 7) \times (2 \times a^*)$?

(c) Find a natural number b that satisfies (3). In other words: $b \equiv 3 \pmod{5}$ and is a multiple of 3 and 7.

(d) Find a natural number c that satisfies (4). That is, c is a multiple of 3 and 5 and $\equiv 4 \pmod{7}$.

(e) Putting together your answers for Part (a), (b), (c) and (d), report an x that indeed satisfies (1).

Solution:

(a) Observe that $a+b+c \equiv 2+0+0 \pmod{3}$, $a+b+c \equiv 0+3+0 \pmod{5}$ and $a+b+c \equiv 0+0+4 \pmod{7}$. Therefore $x = a+b+c$ indeed satisfies the conditions in (1).

(b) This question asks to find a number $0 \leq a < 3 \times 5 \times 7$ that is divisible by 5 and 7 and returns 2 when divided by 3. Let's first look at Part (b.i):

   (b.i) Observe that $(5 \times 7) \equiv 35 \equiv 2 \pmod{3}$. Multiplying both sides by 2, this means that $2 \times (5 \times 7) \equiv 4 \pmod{3} \equiv 1 \pmod{3}$. So, the multiplicative inverse of 5×7, a^* is exactly 2. To verify this: observe that $(5 \times 7) \times 2 = 70 = 3 \times 23 + 1$. Therefore $(5 \times 7) \times 2 \equiv 1 \pmod{3}$.

   Consider $5 \times 7 \times a^*$. Since it is a multiple of 5 and 7, it is equal to 0 modulo either of these numbers. On the other hand, $5 \times 7 \times a^* \equiv 1 \pmod{3}$, since a^* is precisely defined to be the multiplicative inverse of 5×7 modulo 3.

   Consider $5 \times 7 \times (2 \times a^*) = 140$. It is a multiple of, and is therefore 0 modulo both 5 and 7. On the other hand, $5 \times 7 \times (2 \times a^*) \equiv 1 \times 2 \pmod{3}$, for the same reason that a^* is defined to be the multiplicative inverse of 5×7 modulo 3.

   Indeed observe that $5 \times 7 \times (2 \times a^*) = 140$ precisely satisfies the criteria required in Part (b). It is equivalent to 0 modulo 5 and 7 and $\equiv 2 \pmod{3}$.

(c) Let's try to use a similar approach as Part (b). In particular, first observe that $3 \times 7 \equiv 21 \equiv 1 \pmod{5}$. Therefore, b^*, the multiplicative inverse of 3×7 modulo 5 is in fact 1! So, let us consider $3 \times 7 \times (3 \times b^*) = 63$: this is a multiple of 3 and 7 and is therefore 0 modulo both these numbers. On the other hand, $3 \times 7 \times (3 \times b^*) \equiv 3 \pmod{5}$ for the reason that b^* is the multiplicative inverse of 3×7 modulo 5.

(d) Yet again the approach of Part (b) proves to be useful! Observe that $3 \times 5 \equiv 15 \equiv 1 \pmod{7}$. Therefore, c^*, the multiplicative inverse of 3×5 modulo 7 turns out to be 1. So, let us consider $3 \times 5 \times (4 \times c^*) = 60$: this is a multiple of 3 and 5. is therefore 0 modulo both these numbers. On the other hand, $3 \times 5 \times (4 \times c^*) \equiv 4 \pmod{7}$ for the reason that c^* is the multiplicative inverse of 3×5 modulo 7.

(e) From Parts (b), (c) and (d) we find a choice of a, b, c (respectively $= 140, 63, 60$) which satisfues (2), (3) and (4). Together with Part (a) of the question, this implies that $x = a + b + c = 263$ satisfies the required criterion in (1).

To verify this: observe that,

$$263 = 87 \times 3 + 2,$$
$$263 = 52 \times 5 + 3,$$
$$263 = 37 \times 7 + 4.$$