1 Short Answers

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

2 Always, Sometimes, or Never

In each part below, you are given some information about the so-called original graph, OG. Using only the information in the current part, say whether OG will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) OG can be vertex-colored with 4 colors.

(b) OG requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of OG and v is the number of vertices of OG.

(d) OG is connected, and each vertex in OG has degree at most 2.

(e) Each vertex in OG has degree at most 2.

3 Trees and Components

(a) Bob removed a degree 3 node from an n-vertex tree. How many connected components are there in the resulting graph? Please provide an explanation.

(b) Given an n-vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph? Please provide an explanation.

4 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.