1 Short Answers

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?
(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

2 Always, Sometimes, or Never

In each part below, you are given some information about a graph \( G \). Using only the information in the current part, say whether \( G \) will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) \( G \) can be vertex-colored with 4 colors.
(b) \( G \) requires 7 colors to be vertex-colored.
(c) \( e \leq 3v - 6 \), where \( e \) is the number of edges of \( G \) and \( v \) is the number of vertices of \( G \).
(d) \( G \) is connected, and each vertex in \( G \) has degree at most 2.
(e) Each vertex in \( G \) has degree at most 2.
3 Hypercubes

The vertex set of the \( n \)-dimensional hypercube \( G = (V, E) \) is given by \( V = \{0, 1\}^n \) (recall that \( \{0, 1\}^n \) denotes the set of all \( n \)-bit strings). There is an edge between two vertices \( x \) and \( y \) if and only if \( x \) and \( y \) differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an \( n \)-dimensional hypercube can be colored using \( n \) colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any \( n \geq 1 \), the \( n \)-dimensional hypercube is bipartite.

4 Triangular Faces

Suppose we have a connected planar graph \( G \) with \( v \) vertices and \( e \) edges such that \( e = 3v - 6 \). Prove that in any planar drawing of \( G \), every face must be a triangle; that is, prove that every face must be incident to exactly three edges of \( G \).