CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate

DIS 3C

1 Countability Intro

Note 11

A function $f : A \to B$ maps elements from set A to set B.

f is *injective* if it maps distinct elements to distinct elements, and *surjective* if every element in B is mapped to by some element in A. If f is both injective and surjective, it is *bijective*, and the sets A and B are said to have the same *cardinality* (size). The cardinality of a set is denoted by $|A|$.

f is bijective if and only if there exists an inverse function $f^{-1} : B \to A$ such that $f^{-1}(f(a)) = a$ for all $a \in A$ and $f(f^{-1}(b)) = b$ for all $b \in B$.

Countability: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)
- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \leq |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \leq |S|$; together, we have $|\mathbb{N}| = |S|$.
- *Uncountably infinite*: unable to be listed out

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0,1]$:

$\mathbb{N}$	$[0,1]$
0	0 . ⑦ 3 2 0 5 0 ⋯
1	0 . 4 ① 4 2 1 3 ⋯
2	0 . 6 1 ⑧ 0 3 3 ⋯
3	0 . 1 8 2 ⑧ 1 8 ⋯
4	0 . 1 4 1 5 ⑨ 2 ⋯
5	0 . 5 7 7 2 1 ⑤ ⋯
⋮	⋮
?	0 . 8 2 9 9 1 6 ⋯

  If we change the digits along the diagonal, the new decimal created is different from every single element in the list in at least one place, so it's not in the list—this is a contradiction.

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0,1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

0	$\cdots$	0	0	0	0	$\textcircled{1}$
1	$\cdots$	0	1	2	$\textcircled{3}$	4
2	$\cdots$	5	2	$\textcircled{8}$	2	3
3	$\cdots$	9	$\textcircled{4}$	3	2	1
$\vdots$				$\vdots$		
?	$\cdots$	1	5	9	4	2

What is wrong with this argument?

2 Count It!

Note 11

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{A}$.

(f) The set of infinite-length strings over the English alphabet.

3 Counting Cartesian Products

Note 11

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

4 Counting Functions

Note 11

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.