1 Farmer’s Market

Suppose you want \( k \) items from the farmer’s market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

(b) There are pumpkins, apples, oranges, and pears at the market.

(c) There are \( n \) kinds of fruits at the market, and you want to end up with at least two different types of fruit.

2 Inclusion and Exclusion

What is the total number of positive integers strictly less than 100 that are also coprime to 100?
Edward, one of the previous head TA’s, has been hard at work on his latest project, \textit{CS70: The Musical}. It’s now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$  

(b) Edward would now like to select a crew out of $n$ people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal’s Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$
(c) There are \( n \) actors lined up outside of Edward’s office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

\[
\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}
\]

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

\[
\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.
\]