CS 70 Discrete Mathematics and Probability Theory
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# DIS 4C

## 1 Countability and the Halting Problem

Prove the Halting Problem using the set of all programs and inputs.

a) What is a reasonable representation for a computer program? Using this definition, show that the set of all programs are countable. *(Hint: Python Code)*

b) We consider only finite-length inputs. Show that the set of all inputs are countable.

c) Assume that you have a program that tells you whether or not a given program halts on a specific input. Since the set of all programs and the set of all inputs are countable, we can enumerate them and construct the following table.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | ... |
|---|---|---|---|---|---|
| $p_1$ | H | L | H | L | ... |
| $p_2$ | L | L | L | H | ... |
| $p_3$ | H | L | H | L | ... |
| $p_4$ | L | H | L | L | ... |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

An $H$ (resp. $L$) in the $i$th row and $j$th column means that program $p_i$ halts (resp. loops) on input $x_j$. Now write a program that is not within the set of programs in the table above.

d) Find a contradiction in part a and part c to show that the halting problem can't be solved.

## 2 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program $P$ on input $x$ prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program $P$ prints "Hello World!" before running the $k$th line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program $P$ prints "Hello World!" in the first $k$ steps of its execution. Is there a computer program that can perform this task? Justify your answer.

# 3 Fixed Points

Consider the problem of determining if a function $F$ has any fixed points. That is, given a function $F$ that takes inputs from some (possibly infinite) set $\mathscr{X}$, we want to know if there is any input $x \in \mathscr{X}$ such that $F(x)$ outputs $x$. Prove that this problem is undecidable.