Combinations of Events Intro

Note 14

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is just a generalization of the definition of conditional probability: $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]$

Union Bound: Derived from the principle of inclusion-exclusion, the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \le \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$

$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \le \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

1 Probability Potpourri

Note 13
Note 14

Provide brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}[A \setminus B] \ge \mathbb{P}[A] - \mathbb{P}[B]$.

(b) Suppose $\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \overline{C}]$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(c) If A and B are disjoint, does that imply they're independent?

2 Balls and Bins

Note 14

Suppose you throw b balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Let A be the event that at least k bins are empty. Let m be the number of subsets of k bins out of the total n bins. If we assume A_i is the event that the ith subset of k bins is empty. Then we can write A as the union of A_i's:

$$A = \bigcup_{i=1}^{m} A_i.$$

Compute m in terms of n and k, and use the union bound to give an upper bound on the probability $\mathbb{P}[A]$.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

3 Birthdays

Note 14

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that after the first 3 people's birthdays are recorded, no match has occurred (i.e. each person has a unique birthday)?

(b) What is the probability that the first 3 people all share the same birthday?

(c) What is the probability that it takes more than 20 people for a match to occur?

(d) What is the probability that it takes exactly 20 people for a match to occur?

(e) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?