1 Head Count

Consider a coin with $P[\text{Heads}] = 2/5$. Suppose you flip the coin 20 times, and define $X$ to be the number of heads.

(a) What is $P[X = k]$, for some $0 \leq k \leq 20$?

(b) Name the distribution of $X$ and what its parameters are.

(c) What is $P[X \geq 1]$? Hint: You should be able to do this without a summation.

(d) What is $P[12 \leq X \leq 14]$?

Solution:

(a) There are a total of $\binom{20}{k}$ ways to select $k$ coins to be heads. The probability that the selected $k$ coins to be heads is $(\frac{2}{5})^k$, and the probability that the rest are tails is $(\frac{3}{5})^{20-k}$. Putting this together, we have

$$P[X = k] = \binom{20}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{20-k}.$$

(b) Since we have 20 independent trials, with each trial having a probability 2/5 of success, $X \sim \text{Binomial}(20, 2/5)$.

(c)

$$P[X \geq 1] = 1 - P[X = 0] = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$P[12 \leq X \leq 14] = P[X = 12] + P[X = 13] + P[X = 14]$$

$$= \binom{20}{12} \left(\frac{2}{5}\right)^{12} \left(\frac{3}{5}\right)^8 + \binom{20}{13} \left(\frac{2}{5}\right)^{13} \left(\frac{3}{5}\right)^7 + \binom{20}{14} \left(\frac{2}{5}\right)^{14} \left(\frac{3}{5}\right)^6.$$
2 Head Count II

Consider a coin with \( P[\text{Heads}] = \frac{3}{4} \). Suppose you flip the coin until you see heads for the first time, and define \( X \) to be the number of times you flipped the coin.

(a) What is \( P[X = k] \), for some \( k \geq 1 \)?

(b) Name the distribution of \( X \) and what its parameters are.

(c) What is \( P[X > k] \), for some \( k \geq 0 \)?

(d) What is \( P[X < k] \), for some \( k \geq 1 \)?

(e) What is \( P[X > k \mid X > m] \), for some \( k \geq m \geq 0 \)? How does this relate to \( P[X > k - m] \)?

Solution:

(a) If we flipped \( k \) times, then we had \( k - 1 \) tails and 1 head, in that order, giving us

\[
P[X = k] = \frac{3}{4} \left( 1 - \frac{3}{4} \right)^{k-1} \left( \frac{1}{4} \right)^{k-1}.
\]

(b) \( X \sim \text{Geometric} \left( \frac{3}{4} \right) \)

(c) If we had to flip more than \( k \) times before seeing our first heads, then our first \( k \) flips must have been tails, giving us

\[
P[X > k] = \left( 1 - \frac{3}{4} \right)^k = \left( \frac{1}{4} \right)^k.
\]

(d) Notice \( P[X < k] = 1 - P[X \geq k] = 1 - P[X > k - 1] \) since \( X \) can only take on integer values. Along similar lines to the previous part, we then have

\[
P[X < k] = 1 - P[X > k - 1] = 1 - \left( 1 - \frac{3}{4} \right)^{k-1} = 1 - \left( \frac{1}{4} \right)^{k-1}.
\]

(e) By part (c), we have

\[
P[X > k \mid X > m] = \frac{P[X > k \cap X > m]}{P[X > m]} = \frac{P[X > k]}{P[X > m]} = \left( \frac{1}{4} \right)^{k-m}.
\]

However, note that this is exactly \( P[X > k - m] \). The reason this makes sense is that if we want to compute the probability that the first heads occurs after \( k \) flips, and we know that the first heads occurs after \( m \) flips, then the first \( m \) flips are tails. Thus, by the independence of the coin flips, the first \( m \) flips don’t matter, and so we only need to compute the probability that the first heads occurs after \( k - m \) flips. This is called the memorylessness property of the geometric distribution.
3 Shuttles and Taxis at Airport

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson distribution. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

(a) What is the distribution of the following:

(i) The number of taxis that arrive between times 00:00 and 00:20?
(ii) The number of shuttles that arrive between times 00:00 and 00:20?
(iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?

(b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

(c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?

(d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

Solution:

(a) (i) Let $T([0, 20])$ denote the number of taxis that arrive between times 00:00 and 00:20. This interval has length 20 minutes, so the number of taxis $T([0, 20])$ arriving in this interval is distributed according to Poisson($\lambda_2 \cdot 20$) = Poisson(2), i.e.

$$\mathbb{P}[T([0, 20]) = t] = \frac{2^t e^{-2}}{t!}, \text{ for } t = 0, 1, 2, \ldots.$$ 

(ii) Let $S([0, 20])$ denote the number of shuttles that arrive between times 00:00 and 00:20. This interval has length 20 minutes, so the number of shuttles $S([0, 20])$ arriving in this interval is distributed according to Poisson($\lambda_1 \cdot 20$) = Poisson(1), i.e.

$$\mathbb{P}[S([0, 20]) = s] = \frac{1^s e^{-1}}{s!}, \text{ for } s = 0, 1, 2, \ldots.$$ 

(iii) Let $N([0, 20]) = S([0, 20]) + T([0, 20])$ denote the total number of pickup vehicles (taxis and shuttles) arriving between times 00:00 and 00:20. Since the sum of independent Poisson random variables is Poisson distributed with parameter given by the sum of the individual parameters, we have $N([0, 20]) \sim$ Poisson(3), i.e.

$$\mathbb{P}[N([0, 20]) = n] = \frac{3^n e^{-3}}{n!}, \text{ for } n = 0, 1, 2, \ldots.$$
(b) We have

\[ \mathbb{P}[T([0, 20])] = 3 = \frac{2^3 e^{-2}}{3!} \quad \text{and} \quad \mathbb{P}[S([0, 20])] = 1 = \frac{1^0 e^{-1}}{1!}. \]

Since the taxis and the shuttles arrive independently, the probability that exactly 3 taxis and 1 shuttle arrive in this interval is given by the product of their individual probabilities, i.e.

\[ \frac{2^3 e^{-2}}{3!} \cdot \frac{1^0 e^{-1}}{1!} = \frac{4}{3} e^{-3} \approx 0.0664. \]

(c) Let \( A \) be the event that exactly 1 taxi arrives between times 00:00 and 00:20. Let \( B \) be the event that exactly 1 vehicle arrives between times 00:00 and 00:20. We have

\[ \mathbb{P}[B] = \frac{3^1 e^{-3}}{1!}. \]

Event \( A \cap B \) is the event that exactly 1 taxi and 0 shuttles arrive between times 00:00 and 00:20. Hence

\[ \mathbb{P}[A \cap B] = \frac{2^1 e^{-2} \cdot 1^0 e^{-1}}{1!} \cdot 0! = \frac{2}{3} e^{-3}. \]

Thus, we get

\[ \mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{2}{3}. \]

(d) The event that you need to wait for more than 10 minutes starting 00:20 is equivalent to the event that no vehicle arrives between times 00:20 and 00:30. Let \( N([20, 30]) \) denote the number of vehicles that arrive between times 00:20 and 00:30. This interval has length 10 minutes, so \( N([20, 30]) \sim \text{Poisson}(\lambda_1 + \lambda_2 \cdot 10) = \text{Poisson}(3/2) \). Since Poisson arrivals in disjoint intervals are independent, we have

\[ \mathbb{P}[N([20, 30]) = 0 | T([0, 20]) = 3, S([0, 20]) = 1] = \mathbb{P}[N([20, 30]) = 0] \sim \frac{1.5^0 e^{-1.5}}{0!} = e^{-1.5} \approx 0.2231. \]