

Graph Theory

Note 10

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E , where each edge joins a pair of vertices. Edges are unordered pairs $\{u, v\}$ in an undirected graph and ordered pairs (u, v) , directed from u to v , in a directed graph. Unless otherwise specified, graphs are undirected and simple.

Degree: The degree of a vertex v is the number of edges incident to it, denoted $\deg(v)$.

Degree-sum formula: $\sum_{v \in V} \deg(v) = 2|E|$, since each edge is incident to 2 vertices.

Walk: A sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$, possibly with repeated vertices.

Path: A walk with no repeated vertices.

Tour: A walk that starts and ends at the same vertex.

Cycle: A tour with no repeated vertices other than the start and end.

Eulerian tour: A tour that uses every edge in the graph exactly once.

Connected: A graph is connected if there is a path between every pair of vertices.

Complete graph: The complete graph on n vertices, denoted K_n , has an edge between every pair of vertices.

Bipartite graph: A graph whose vertices can be split into two groups so that every edge goes between the groups.

Tree: A graph satisfying any of the following equivalent conditions:

- connected and acyclic
- connected and has $|V| - 1$ edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

Hypercube: The hypercube of dimension n has 2^n vertices, each labeled by a length- n bit string. Edges connect vertices that differ by exactly one bit. A hypercube of dimension $n + 1$ can be recursively constructed by creating two copies of an n -dimensional hypercube and connecting corresponding vertices by an edge.

Planar graph: A graph that has at least one drawing in the plane with no edge crossings (other drawings may have crossings). Such a drawing divides the plane into regions called faces.

A connected planar graph G with v vertices, e edges, and f faces satisfies:

- Euler's formula: $v + f = e + 2$.
- If $v \geq 3$, then $e \leq 3v - 6$.

A graph is non-planar iff it contains K_5 or $K_{3,3}$ (the complete bipartite graph with 3 vertices on each side).

1 Short Graph Proofs

Note 10

- (a) Prove that a tree with n vertices has exactly $n - 1$ edges.
- (b) Prove that every connected component in an acyclic graph is a tree.
- (c) Let $G = (V, E)$ be an acyclic graph with k connected components. Prove that $|E| = |V| - k$.
- (d) Let $G = (V, E)$ be a graph with $|E| \geq |V|$. Prove that G contains a cycle.
- (e) Prove that if G is a connected planar graph with $v \geq 3$ vertices and e edges, then $e \leq 3v - 6$.
- (f) Prove that $K_{3,3}$ is nonplanar.

Solution:

- (a) We use induction on n .

Base Case: $n = 1$. A single vertex has 0 edges.

Inductive Hypothesis: Every tree with k vertices has $k - 1$ edges.

Inductive Step: Let T be a tree with $k + 1 \geq 2$ vertices. T has a leaf ℓ (for example, an endpoint of a longest path). Remove ℓ and its one edge. The result is still acyclic, and still connected because a leaf cannot be an interior vertex of any path. So it is a tree with k vertices, and has $k - 1$ edges by the inductive hypothesis. Adding ℓ back adds exactly one edge, so T has k edges.

- (b) A connected component is connected by definition, and it is acyclic because any cycle in it would be a cycle in the graph. So it is a tree.
- (c) Let the components have $n_1, \dots, n_k$ vertices. By part (b) each is a tree, so by part (a) the i -th component has $n_i - 1$ edges. Summing, $|E| = \sum_{i=1}^k (n_i - 1) = |V| - k$.
- (d) Suppose G is acyclic, and let $k \geq 1$ be its number of components. By part (c), $|E| = |V| - k < |V|$, contradicting $|E| \geq |V|$. So G contains a cycle.
- (e) Fix a planar drawing of G with f faces, and let s_i be the number of edges bordering face i . Since G is connected with $v \geq 3$ vertices, every face is bounded by at least 3 edges, so $s_i \geq 3$. Each edge borders at most two faces, so

$$3f \leq \sum_{i=1}^f s_i \leq 2e.$$

Thus $f \leq \frac{2}{3}e$, and Euler's formula gives $2 = v - e + f \leq v - \frac{1}{3}e$, i.e., $e \leq 3v - 6$.

- (f) Suppose $K_{3,3}$ is planar. It is connected with $v = 6$ and $e = 9$, so Euler's formula gives $f = 2 - v + e = 5$. Since $K_{3,3}$ is bipartite, it has no cycles of length 3, so every face is bounded by at least 4 edges. Counting as in part (e), $4f \leq 2e$, i.e., $20 \leq 18$, a contradiction. So $K_{3,3}$ is nonplanar.

2 Build-Up Error?

Note 10

What is wrong with the following “proof”? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

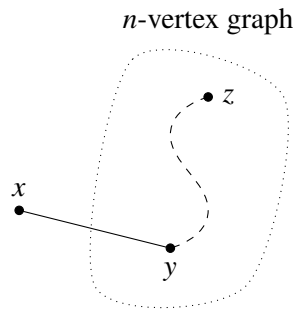
False Claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected.

Proof? We use induction on the number of vertices $n \geq 2$.

Base Case: The only valid graph has two vertices joined by an edge. This graph is connected, so the base case is true.

Induction Hypothesis: Assume the claim is true for some $n \geq 2$.

Inductive Step: We prove the claim is also true for $n + 1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the induction hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n + 1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z . Since x has degree at least 1, there is an edge from x to some other vertex; call it y . Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z . This proves the claim for $n + 1$. $\square$

Solution:

Counterexample: the graph with vertices $\{1, 2, 3, 4\}$ and edges $\{1, 2\}, \{3, 4\}$. Every vertex has degree 1, but the graph is disconnected.

What went wrong: the inductive step starts from an n -vertex graph satisfying the hypothesis and adds a vertex. This proves the claim only for $(n + 1)$ -vertex graphs that can be *built up* this way; it never shows that every $(n + 1)$ -vertex graph with minimum degree 1 arises like this. The counterexample does not: deleting any of its vertices leaves a vertex of degree 0. This is *build-up error*: assuming every graph of size $n + 1$ with a property can be built from a graph of size n with the same property.

How to avoid it: shrink down, then grow back. Start from an *arbitrary* graph G with $n + 1$ vertices, remove a vertex (or edge), apply the inductive hypothesis to the smaller graph, and add the vertex back. Here this fails immediately: removing a vertex from G can leave a vertex of degree 0, so the inductive hypothesis does not apply—as it should, since the claim is false.

3 Hypercubes

Note 10

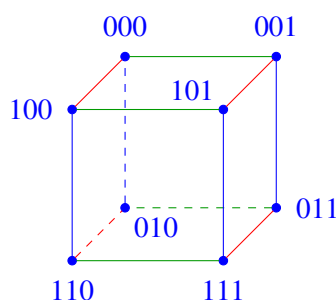
The vertex set of the n -dimensional hypercube $G = (V, E)$ is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all n -bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

- Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.
- Show that the edges of an n -dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

Solution:

- (a) The 1-dimensional hypercube is a single edge 0—1. The 2-dimensional hypercube is a square with vertices 00, 01, 11, 10 in order around it. The 3-dimensional hypercube is a cube: two copies of the square, one with every label prefixed by 0 and one prefixed by 1, with corresponding vertices joined (pictured in part (b) below).
- (b) Color each edge with the position $i \in \{1, \dots, n\}$ of the bit in which its endpoints differ. The n edges at any vertex flip n different bits, so they all get different colors.

The 3-dimensional case is shown below (red: first bit, black: second bit, green: third bit):



Alternate proof (induction). For $n = 1$, the single edge gets one color. The $(n + 1)$ -dimensional hypercube consists of two n -dimensional hypercubes, each edge-colored with colors $1, \dots, n$ by the inductive hypothesis, plus one edge joining each pair of corresponding vertices. These joining edges share no endpoints, so giving them all the new color $n + 1$ yields a valid coloring.

4 Counting Edges

Note 10

- (a) Prove that for all $n \geq 1$, the complete graph K_n has exactly $n(n - 1)/2$ edges.
- (b) Prove that for all $n \geq 1$, the n -dimensional hypercube has exactly $n \cdot 2^{n-1}$ edges.

Solution:

- (a) We use induction on n .

Base Case: $n = 1$. K_1 has no edges, and $1 \cdot 0/2 = 0$.

Inductive Hypothesis: K_k has $k(k - 1)/2$ edges.

Inductive Step: Let v be any vertex of K_{k+1} . Deleting v leaves K_k , which has $k(k - 1)/2$ edges by the inductive hypothesis, and v is adjacent to all k other vertices. So K_{k+1} has

$$\frac{k(k - 1)}{2} + k = \frac{k(k + 1)}{2}$$

edges.

Alternate proof. Every vertex of K_n has degree $n - 1$, so by the degree-sum formula $2|E| = n(n - 1)$.

- (b) We use induction on n .

Base Case: $n = 1$. The 1-dimensional hypercube is a single edge between 0 and 1, and $1 \cdot 2^0 = 1$.

Inductive Hypothesis: The k -dimensional hypercube has $k \cdot 2^{k-1}$ edges.

Inductive Step: Split the vertices of the $(k+1)$ -dimensional hypercube by their first bit.

- Within each half, two strings are adjacent exactly when their last k bits differ in one position, so each half is a k -dimensional hypercube with $k \cdot 2^{k-1}$ edges by the inductive hypothesis.
- Across the halves, two strings are adjacent exactly when their last k bits agree, so each of the 2^k vertices in one half has exactly one neighbor in the other, giving 2^k edges.

In total there are $2 \cdot k \cdot 2^{k-1} + 2^k = (k+1) \cdot 2^k$ edges.

Alternate proof. Each of the 2^n vertices has degree n (one neighbor per bit), so by the degree-sum formula $2|E| = n \cdot 2^n$, i.e., $|E| = n \cdot 2^{n-1}$.

5 Proofs in Graphs

Note 10

- (a) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and no walk uses an edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof. Also, note that Euler's Theorem holds for both simple graphs and multigraphs (graphs where multiple edges between one pair of vertices is permitted).]

- (b) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of the shortest paths from some fixed vertex to all the other vertices.]

Solution:

- (a) Pair up the $2m$ odd-degree vertices arbitrarily and add an edge between each pair, allowing multi-edges. Every vertex now has even degree and the graph is still connected, so by Euler's theorem it has an Eulerian tour. Each vertex is an endpoint of at most one added edge, so no two added edges are consecutive in the tour. Deleting the m added edges therefore cuts the tour into exactly m nonempty walks, which together use every edge of G exactly once.

- (b) We prove both directions.

($\Rightarrow$) Suppose G is bipartite with parts L and R , and consider any tour, starting at a vertex v_0 ; by symmetry, say $v_0 \in L$. Every edge goes between L and R , so after an odd number of edges the tour is at a vertex in R , and after an even number of edges it is at a vertex in L . The tour ends back at $v_0 \in L$, so its length is even.

($\Leftarrow$) Suppose G has no tours of odd length. It suffices to show that each connected component is bipartite, so assume G is connected. Fix a vertex v and partition the vertices by the parity of their distance to v :

$$R = \{u \mid \text{the length of a shortest path from } u \text{ to } v \text{ is even}\}$$

$$L = \{u \mid \text{the length of a shortest path from } u \text{ to } v \text{ is odd}\}$$

We claim no edge joins two vertices in the same set. Suppose u_1, u_2 are adjacent and lie in the same set. Then the shortest paths from v to u_1 and from u_2 to v have the same parity, so the closed walk consisting of a shortest path from v to u_1 , the edge $\{u_1, u_2\}$, and a shortest path from u_2 to v has odd length. This is a tour of odd length, a contradiction. Hence every edge goes between L and R , and G is bipartite.

6 Not Everything is Normal: Odd-Degree Vertices

Note 10

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(a) Direct proof (e.g., counting the number of edges in G).

[Hint: In lecture, we proved that $\sum_{v \in V} \deg(v) = 2|E|$.]

(b) Induction on $m = |E|$ (number of edges).

(c) Induction on $n = |V|$ (number of vertices).

Solution: Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(a) Observe that

$$\sum_{v \in V} \deg(v) = 2|E|$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd-degree vertices $V_{\text{odd}}(G)$ and the even-degree vertices, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} \deg(v) = 2|E| - \sum_{v \notin V_{\text{odd}}(G)} \deg(v).$$

Both terms on the right-hand side are even ($2|E|$ is even, and each $\deg(v)$ in the second sum is even). So the left-hand side is even. But each term on the left-hand side is odd, so there must be an even number of terms. Therefore $|V_{\text{odd}}(G)|$ is even.

(b) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G , then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m + 1$ edges. Remove an arbitrary edge $\{u, v\}$ from G , so the resulting graph G' has m edges. By the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u, v\}$ back. This increases $\deg(u)$ and $\deg(v)$ by one and changes no other degree, so u and v each switch between odd and even degree while every other vertex stays put. Hence $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$ (both leave V_{odd} , both join, or one of each), and $|V_{\text{odd}}(G)|$ is even.

(c) We use induction on $n \geq 1$.

Base case $n = 1$: If G has only 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n + 1$ vertices. Remove a vertex v and all edges incident to it. The resulting graph G' has n vertices, so by the inductive hypothesis $|V_{\text{odd}}(G')|$ is even. Now add v and its edges back, and let N_v be the set of neighbors of v . Each neighbor gains one edge, so its degree parity flips: it either joins or leaves V_{odd} , changing the count by ± 1 . The vertex v itself has degree $|N_v|$, so it joins V_{odd} exactly when $|N_v|$ is odd. Therefore

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + s + t,$$

where s is a sum of $|N_v|$ terms each equal to ± 1 , and $t = 1$ if $|N_v|$ is odd and $t = 0$ otherwise. Both s and t have the same parity as $|N_v|$, so $s + t$ is even. Since $|V_{\text{odd}}(G')|$ is even, $|V_{\text{odd}}(G)|$ is even.

Note how this proof is more complicated than the proof in part (b), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.