

Graph Theory

Note 10

A graph $G = (V, E)$ consists of a set of vertices V and a set of edges E , where each edge joins a pair of vertices. Edges are unordered pairs $\{u, v\}$ in an undirected graph and ordered pairs (u, v) , directed from u to v , in a directed graph. Unless otherwise specified, graphs are undirected and simple.

Degree: The degree of a vertex v is the number of edges incident to it, denoted $\deg(v)$.

Degree-sum formula: $\sum_{v \in V} \deg(v) = 2|E|$, since each edge is incident to 2 vertices.

Walk: A sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$, possibly with repeated vertices.

Path: A walk with no repeated vertices.

Tour: A walk that starts and ends at the same vertex.

Cycle: A tour with no repeated vertices other than the start and end.

Eulerian tour: A tour that uses every edge in the graph exactly once.

Connected: A graph is connected if there is a path between every pair of vertices.

Complete graph: The complete graph on n vertices, denoted K_n , has an edge between every pair of vertices.

Bipartite graph: A graph whose vertices can be split into two groups so that every edge goes between the groups.

Tree: A graph satisfying any of the following equivalent conditions:

- connected and acyclic
- connected and has $|V| - 1$ edges
- connected, and removing any edge disconnects the graph
- acyclic, and adding any edge creates a cycle

Hypercube: The hypercube of dimension n has 2^n vertices, each labeled by a length- n bit string. Edges connect vertices that differ by exactly one bit. A hypercube of dimension $n + 1$ can be recursively constructed by creating two copies of an n -dimensional hypercube and connecting corresponding vertices by an edge.

Planar graph: A graph that has at least one drawing in the plane with no edge crossings (other drawings may have crossings). Such a drawing divides the plane into regions called faces.

A connected planar graph G with v vertices, e edges, and f faces satisfies:

- Euler's formula: $v + f = e + 2$.
- If $v \geq 3$, then $e \leq 3v - 6$.

A graph is non-planar iff it contains K_5 or $K_{3,3}$ (the complete bipartite graph with 3 vertices on each side).

1 Short Graph Proofs

Note 10

(a) Prove that a tree with n vertices has exactly $n - 1$ edges.

(b) Prove that every connected component in an acyclic graph is a tree.

(c) Let $G = (V, E)$ be an acyclic graph with k connected components. Prove that $|E| = |V| - k$.

(d) Let $G = (V, E)$ be a graph with $|E| \geq |V|$. Prove that G contains a cycle.

(e) Prove that if G is a connected planar graph with $v \geq 3$ vertices and e edges, then $e \leq 3v - 6$.

(f) Prove that $K_{3,3}$ is nonplanar.

2 Build-Up Error?

Note 10 What is wrong with the following “proof”? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

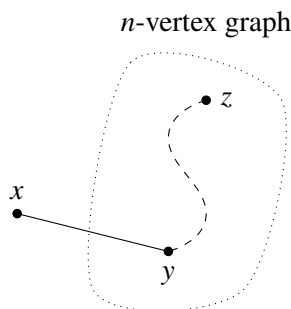
False Claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected.

Proof? We use induction on the number of vertices $n \geq 2$.

Base Case: The only valid graph has two vertices joined by an edge. This graph is connected, so the base case is true.

Induction Hypothesis: Assume the claim is true for some $n \geq 2$.

Inductive Step: We prove the claim is also true for $n + 1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the induction hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n + 1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z . Since x has degree at least 1, there is an edge from x to some other vertex; call it y . Thus, we can obtain a path from x to z by adjoining the edge $\{x, y\}$ to the path from y to z . This proves the claim for $n + 1$. $\square$

3 Hypercubes

Note 10

The vertex set of the n -dimensional hypercube $G = (V, E)$ is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all n -bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an n -dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

4 Counting Edges

Note 10

(a) Prove that for all $n \geq 1$, the complete graph K_n has exactly $n(n-1)/2$ edges.

(b) Prove that for all $n \geq 1$, the n -dimensional hypercube has exactly $n \cdot 2^{n-1}$ edges.

5 Proofs in Graphs

Note 10

- (a) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and no walk uses an edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof. Also, note that Euler's Theorem holds for both simple graphs and multigraphs (graphs where multiple edges between one pair of vertices is permitted).]

- (b) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of the shortest paths from some fixed vertex to all the other vertices.]

6 Not Everything is Normal: Odd-Degree Vertices

Note 10

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

- (a) Direct proof (e.g., counting the number of edges in G).

[*Hint:* In lecture, we proved that $\sum_{v \in V} \deg(v) = 2|E|$.]

- (b) Induction on $m = |E|$ (number of edges).

- (c) Induction on $n = |V|$ (number of vertices).