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1 Head Count

Consider a coin with $\mathbb{P}[\text{Heads}] = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is $\mathbb{P}[X = k]$, for some $0 \le k \le 20$?

(b) Name the distribution of X and what its parameters are.

(c) What is $\mathbb{P}[X \ge 1]$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}[12 \le X \le 14]$?

Solution:

(a) There are a total of $\binom{20}{k}$ ways to select k coins to be heads. The probability that the selected k coins to be heads is $(\frac{2}{5})^k$, and the probability that the rest are tails is $(\frac{3}{5})^{20-k}$. Putting this together, we have

$$\mathbb{P}[X = k] = \binom{20}{k}\left(\frac{2}{5}\right)^k\left(\frac{3}{5}\right)^{20-k}.$$

(b) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(c)

$$\mathbb{P}[X \ge 1] = 1 - \mathbb{P}[X = 0] = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}[12 \le X \le 14] &= \mathbb{P}[X = 12] + \mathbb{P}[X = 13] + \mathbb{P}[X = 14] \\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$

2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C. Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$
$G = 0$			
$G = 1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G = 0)$	
$\mathbb{P}(G = 1)$	

$\mathbb{P}(C = 1)$	$\mathbb{P}(C = 2)$	$\mathbb{P}(C = 3)$

(d) Are G and C independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution:

(a) The sample space is the set of all possible sequences of children that the Browns can have: $\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:

$$\mathbb{P}[g] = \frac{1}{2}$$
$$\mathbb{P}[bg] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
$$\mathbb{P}[bbg] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$\mathbb{P}[bbb] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

(b)

	$C = 1$	$C = 2$	$C = 3$
$G = 0$	0	0	$\mathbb{P}[bbb] = 1/8$
$G = 1$	$\mathbb{P}[g] = 1/2$	$\mathbb{P}[bg] = 1/4$	$\mathbb{P}[bbg] = 1/8$

(c) Marginal distribution for G:

$$\mathbb{P}[G=0] = 0+0+\frac{1}{8} = \frac{1}{8}$$
$$\mathbb{P}[G=1] = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8}$$

Marginal distribution for C:

$$\mathbb{P}[C=1] = 0+\frac{1}{2} = \frac{1}{2}$$
$$\mathbb{P}[C=2] = 0+\frac{1}{4} = \frac{1}{4}$$
$$\mathbb{P}[C=3] = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}$$

(d) No, G and C are not independent. If two random variables are independent, then

$$\mathbb{P}[X=x, Y=y] = \mathbb{P}[X=x]\mathbb{P}[Y=y].$$

To show this dependence, consider an entry in the joint distribution table, such as $\mathbb{P}[G=0, C=3] = 1/8$. This is not equal to $\mathbb{P}[G=0]\mathbb{P}[C=3] = (1/8)\cdot(1/4) = 1/32$, so the random variables are not independent.

(e) We can apply the definition of expectation directly for this problem, since we've computed the marginal distribution for both random variables.

$$\mathbb{E}[G] = 0\cdot\mathbb{P}[G=0]+1\cdot\mathbb{P}[G=1] = 1\cdot\frac{7}{8} = \frac{7}{8}$$
$$\mathbb{E}[C] = 1\cdot\mathbb{P}[C=1]+2\cdot\mathbb{P}[C=2]+3\cdot\mathbb{P}[C=3] = 1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{4} = \frac{7}{4}$$

3 Binary Conditional Probabilities

Let us consider a sample space $\Omega = \{\omega_1, \dots, \omega_N\}$ of size $N > 2$, and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

If for every subset $A \subset \Omega$ of size $|A| = 2$ and every outcome $\omega \in \Omega$ it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$, then is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

Solution: Yes, this is indeed true. To see why, let's take the subset $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \dots, N\}$ and compute: For any $k \in \{1, 2\}$, we have $\mathbb{P}_k(\omega_i \mid A) = \frac{\mathbb{P}_k(\omega_i)}{\mathbb{P}_k(A)}$. Since this expression (by assumption) is the same for $k = 1$ and $k = 2$, we conclude that $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_2(\omega_i)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$. Repeating the reasoning for ω_j, we similarly find that $\frac{\mathbb{P}_1(\omega_j)}{\mathbb{P}_2(\omega_j)} = \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)}$, and whence $\frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)}$. Since this is true for any $i, j \in \{1, \dots, N\}$, we can sum over i to get

$$\frac{1}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_1(\omega_i)}{\mathbb{P}_1(\omega_j)} = \sum_{i=1}^{N} \frac{\mathbb{P}_2(\omega_i)}{\mathbb{P}_2(\omega_j)} = \frac{1}{\mathbb{P}_2(\omega_j)},$$

which shows that $\mathbb{P}_1(\omega_j) = \mathbb{P}_2(\omega_j)$ for all $j \in \{1, \dots, N\}$.