CS 70 Discrete Mathematics and Probability Theory
Fall 2021

DIS 6A

1 Strings

What is the number of strings you can construct given:

(a) n ones, and m zeroes?

(b) n_1 A's, n_2 B's and n_3 C's?

(c) $n_1, n_2, \ldots, n_k$ respectively of k different letters?

Solution:

(a) $\binom{n+m}{n}$

(b) $(n_1 + n_2 + n_3)!/(n_1! \cdot n_2! \cdot n_3!)$

(c) $(n_1 + n_2 + \cdots + n_k)!/(n_1! \cdot n_2! \cdots n_k!)$.

2 The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

Solution:

(a) We use inclusion-exclusion to calculate the number of numbers that satisfy this property. Let A be the set of numbers divisible by 2, B be the set of numbers divisible by 3, and C be the set of numbers divisible by 5. Then, we calculate

$$\begin{aligned}&|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|\\&\quad=\left\lfloor\frac{100}{2}\right\rfloor+\left\lfloor\frac{100}{3}\right\rfloor+\left\lfloor\frac{100}{5}\right\rfloor-\left\lfloor\frac{100}{6}\right\rfloor-\left\lfloor\frac{100}{10}\right\rfloor-\left\lfloor\frac{100}{15}\right\rfloor+\left\lfloor\frac{100}{30}\right\rfloor\\&=50+33+20-16-10-6+3=74\end{aligned}$$

numbers.

(b) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

(c) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

3 Digits

(a) How many 7-digit numbers have no two adjacent digits equal?

(b) How many 5-digit palindromes are there? (A palindrome is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

Solution:

(a) We construct these numbers from left-to-right. We have 9 choices for the first digit (since it cannot be 0). Then, no matter what we choose for the first digit, we have 9 choices for the second digit (any digit except the one that we chose for the first digit). Similarly, for each subsequent digit, we have 9 choices, since each digit can be any of 0 through 9 as long as it does not match the previous digit. Therefore, there are 9 choices for every digit, and hence there are $9^7 = 4,782,969$ such numbers.

(b) We construct the number from left-to-right. We have 9 choices for the first digit (since it can't be 0), then 10 choices for the second digit, then 10 choices for the third digit. But now we're out of choices: the fourth digit must match the second, and the last digit must match the first. Therefore, there are $9 \cdot 10 \cdot 10 = 900$ such numbers.

4 Divisor Graph Colorings

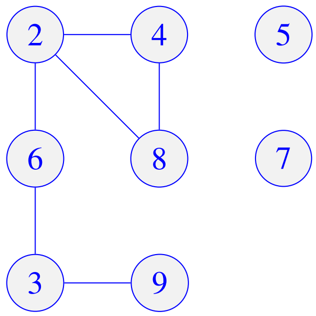
Define G where we have $V = \{2,3,4,5,6,7,8,9\}$, and we add an edge between vertex i and vertex j if i divides j, or j divides i.

(a) Draw G.

(b) Explain why we cannot vertex-color G with only 2 colors.

(c) How many ways can we vertex-color G with 3 colors?

Solution:

(a) 

(b) The vertices 2, 4, and 8 form a length-3 cycle, which cannot be colored.

(c) 432. Vertices 5 and 7 can each take one of three colors. So can vertex 2. Then vertex 4 must take on of two colors, and vertex 8 (being connected to both 2 and 4) is forced to take on a particular color. Vertex 6 (being connected to vertex 2) then has choice of two colors. Vertex 3 (being connected to vertex 6) then has choice of two colors. Vertex 9 (being connected to vertex 3) then has choice of two colors as well. This is a total of $3 \cdot 3 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 432$ colorings.