1 Counting Strings

(a) How many bit strings of length 10 contain at least five consecutive 0's?

(b) How many different ways are there to rearrange the letters of DIAGONALIZATION (15 letters with 3 A's, 3 I's, 2 N's, and 2 O's) without the two N's being adjacent?

Solution:

(a) One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1=4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$. possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.

(b)
$$\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!} = \frac{13 \cdot 14!}{3!3!2!2!}$$

The word DIAGONALIZATION has 15 letters with 3 A's, 3 I's, 2 N's, and 2 O's, so there are $\frac{15!}{3!3!2!2!}$ ways to rearrange the letters in total.

The number of rearrangements where the two N's are adjacent is $\frac{14!}{3!3!2!}$, where we have considered "NN" as a single character. The difference $\frac{15!}{3!3!2!2!} - \frac{14!}{3!3!2!}$ is then equal to the number of rearrangements without the two N's being adjacent.

2 Teams and Leaders

Prove the following identities using a combinatorial proof.

1. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
2. $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Solution:

1. Imagine you are a teacher picking students to be on a team for some competition. You have $2n$ students, n of whom are boys and the other n are girls.

   RHS: This is simply the number of ways you can pick n students to be on the team.
   LHS: We begin by noticing that $\binom{n}{k}^2 = \binom{n}{k} \cdot \binom{n}{n-k}$. This product gives us the number of ways of picking k girls and $n-k$ boys to be on the team. We add up all the products involving anywhere from 0 girls all the way to n girls. This gives us the total number of ways to pick a team of n students.

2. Imagine the same scenario as part (a) except now you have to choose a female team leader amongst the n students on the team.

   RHS: This is the number of ways of picking the team leader multiplied with the number of ways of picking the rest of the team from the remaining students. The product gives the total number of teams with a female leader.
   LHS: We begin similarly by noticing that $k \cdot \binom{n}{k}^2 = k \cdot \binom{n}{k} \cdot \binom{n}{n-k}$. Here as before we are picking k girls and $n-k$ boys to be on the team. However amongst the k girls on the team, we choose one of them to be the team leader. We add up all the products involving anywhere from 1 girls all the way to n girls. This gives us the total number of ways to pick a team of n students with a female leader.

3 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called Pascal's Identity)

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

Solution:

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

(a) Both directors from the group of experienced directors,

(b) Both directors from the group of inexperienced directors, or

(c) One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

4 Countability: True or False

(a) The set of all irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x+y=1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

Solution:

(a) **True.** Proof by contradiction. Suppose the set of irrationals is countable. From Lecture note 10 we know that the set $\mathbb{Q}$ is countable. Since union of two countable sets is countable, this would imply that the set $\mathbb{R}$ is countable. But again from Lecture note 10 we know that this is not true. Contradiction!

(b) **True.** Multiplying both sides of the modular equation by 7 (the multiplicative inverse of 3 with respect to 10) we get $x \equiv 4 \pmod{10}$. The set of all intergers that solve this is $S = \{10k+4 : k \in \mathbb{Z}\}$ and it is clear that the mapping $k \in \mathbb{Z}$ to $10k+4 \in S$ is a bijection. Since the set $\mathbb{Z}$ is countably infinite, the set S is also countably infinite.

(c) **False.** Let $S \in \mathbb{R} \times \mathbb{R}$ denote the set of all real solutions for the given equation. For any $x' \in \mathbb{R}$, the pair $(x', y') \in S$ if and only if $y' = 1 - x'$. Thus $S = \{(x, 1-x) : x \in \mathbb{R}\}$. Besides, the mapping x to $(x, 1-x)$ is a bijection from $\mathbb{R}$ to S. Since $\mathbb{R}$ is uncountable, we have that S is uncountable too.

(d) **True.** Recall that a function $h : A \to B$ is injective iff $a_1 \neq a_2 \implies h(a_1) \neq h(a_2)$ for all $a_1, a_2 \in A$. Let $x_1, x_2 \in X$ be arbitrary such that $x_1 \neq x_2$. Since g is injective, we have $g(x_1) \neq g(x_2)$. Now, since f is injective, we have $f(g(x_1)) \neq g(g(x_2))$. Hence $f \circ g$ is injective.

(e) **False.** Recall that a function $h : A \to B$ is surjective iff $\forall b \in B, \exists a \in A$ such that $h(a) = b$. Let $g : \{0,1\} \to \{0,1\}$ be given by $g(0) = g(1) = 0$. Let $f : \{0,1\} \to \{0,1\}$ be given by $f(0) = 0$ and $f(1) = 1$. Then $f \circ g : \{0,1\} \to \{0,1\}$ is given by $(f \circ g)(0) = (f \circ g)(1) = 0$. Here f is surjective but $f \circ g$ is not surjective.