1 CS70: The Musical

Edward, one of the previous head TA’s, has been hard at work on his latest project, *CS70: The Musical*. It’s now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$ 

(b) Edward would now like to select a crew out of $n$ people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal’s Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$
(c) There are $n$ actors lined up outside of Edward’s office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}.$$
2 Inclusion and Exclusion

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

3 Countability: True or False

(a) The set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers $x$ that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \rightarrow Z$ and $g : X \rightarrow Y$, let their composition $f \circ g : X \rightarrow Z$ be given by $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) $f$ and $g$ are injective (one-to-one) $\implies f \circ g$ is injective (one-to-one).

(e) $f$ is surjective (onto) $\implies f \circ g$ is surjective (onto).
4 Counting Cartesian Products

For two sets $A$ and $B$, define the cartesian product as $A \times B = \{(a, b) : a \in A, b \in B\}$.

(a) Given two countable sets $A$ and $B$, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.