1 Counting Cartesian Products

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

2 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

3 Undecided?

Let us think of a computer as a machine which can be in any of n states $\{s_1, \dots, s_n\}$. The state of a 10 bit computer might for instance be specified by a bit string of length 10, making for a total of 2^{10} states that this computer could be in at any given point in time. An algorithm $\mathscr{A}$ then is a list of k instructions $(i_0, i_2, \dots, i_{k-1})$, where each i_l is a function of a state c that returns another state u and a number j. Executing $\mathscr{A}(x)$ means computing

$$(c_1, j_1) = i_0(x), \qquad (c_2, j_2) = i_{j_1}(c_1), \qquad (c_3, j_3) = i_{j_2}(c_2), \qquad \dots$$

until $j_\ell \geq k$ for some ℓ, at which point the algorithm halts and returns $c_{\ell-1}$.

(a) How many iterations can an algorithm of k instructions perform on an n-state machine (at most) without repeating any computation?

(b) Show that if the algorithm is still running after $2n^2k^2$ iterations, it will loop forever.

(c) Give an algorithm that decides whether an algorithm $\mathscr{A}$ halts on input x or not. Does your contruction contradict the undecidability of the halting problem?

4 Code Reachability

Consider triplets (M, x, L) where

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M is a Java program
x is some input
L is an integer
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and the question of: if we execute $M(x)$, do we ever hit line L?

Prove this problem is undecidable.