1 Counting Cartesian Products

For two sets $A$ and $B$, define the cartesian product as $A \times B = \{(a, b) : a \in A, b \in B\}$.

(a) Given two countable sets $A$ and $B$, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that $A_1 \times A_2 \times \cdots \times A_n$ is countable.

2 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f$ is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$. 
(b) The set of all functions $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f$ is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

3 Undecided?

Let us think of a computer as a machine which can be in any of $n$ states $\{s_1, \ldots, s_n\}$. The state of a 10 bit computer might for instance be specified by a bit string of length 10, making for a total of $2^{10}$ states that this computer could be in at any given point in time. An algorithm $\mathcal{A}$ then is a list of $k$ instructions $(i_0, i_2, \ldots, i_{k-1})$, where each $i_l$ is a function of a state $c$ that returns another state $u$ and a number $j$. Executing $\mathcal{A}(x)$ means computing

\[(c_1, j_1) = i_0(x), \quad (c_2, j_2) = i_{j_1}(c_1), \quad (c_3, j_3) = i_{j_2}(c_2), \quad \ldots\]

until $j_\ell \geq k$ for some $\ell$, at which point the algorithm halts and returns $c_{\ell-1}$.

(a) How many iterations can an algorithm of $k$ instructions perform on an $n$-state machine (at most) without repeating any computation?
(b) Show that if the algorithm is still running after $2n^2k^2$ iterations, it will loop forever.

(c) Give an algorithm that decides whether an algorithm $A$ halts on input $x$ or not. Does your construction contradict the undecidability of the halting problem?

4 Code Reachability

Consider triplets $(M, x, L)$ where

- $M$ is a Java program
- $x$ is some input
- $L$ is an integer

and the question of: if we execute $M(x)$, do we ever hit line $L$?

Prove this problem is undecidable.