1  Countability: True or False

(a) The set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.
(b) The set of integers $x$ that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.
(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) $f$ and $g$ are injective (one-to-one) $\implies f \circ g$ is injective (one-to-one).
(e) $f$ is surjective (onto) $\implies f \circ g$ is surjective (onto).

2  Counting Cartesian Products

For two sets $A$ and $B$, define the cartesian product as $A \times B = \{(a, b) : a \in A, b \in B\}$.

(a) Given two countable sets $A$ and $B$, prove that $A \times B$ is countable.
(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.
3 Hello World!

Determine the computability of the following tasks. If it’s not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program $P$ on input $x$ prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program $P$ prints "Hello World!" before running the $k$th line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program $P$ prints "Hello World!" in the first $k$ steps of its execution. Is there a computer program that can perform this task? Justify your answer.
4 Code Reachability

Consider triplets \((M, x, L)\) where

- \(M\) is a Java program
- \(x\) is some input
- \(L\) is an integer

and the question of: if we execute \(M(x)\), do we ever hit line \(L\)?

Prove this problem is undecidable.