CS 70 Discrete Mathematics and Probability Theory
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1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

Solution:

(a) Integrate:

$$\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = c\int_1^2 x^{-2}\,\mathrm{d}x = -cx^{-1}\Big|_{x=1}^{2} = -c\Big(\frac{1}{2} - 1\Big) = \frac{c}{2} = 1$$

so $c = 2$.

(b) To find $\mathbb{P}(A)$,

$$\mathbb{P}(A) = \int_{1.5}^{2} f_X(x)\,\mathrm{d}x = 2\int_{1.5}^{2} x^{-2}\,\mathrm{d}x = -2x^{-1}\Big|_{x=1.5}^{2} = -2\Big(\frac{1}{2} - \frac{2}{3}\Big) = \frac{1}{3}.$$

The conditional PDF is thus

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(A)} = 6x^{-2}, \qquad x \in [1.5, 2].$$

2 Joint Practice

Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} Ax^2y^2 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a positive constant.

(a) What is the value of A?

(b) What is the marginal density of X?

(c) What is $\text{cov}(X,Y)$?

Solution:

(a) Since $f_{X,Y}$ is a joint density, we know that it must integrate to 1. Since $f_{X,Y}$ is only nonzero on the unit square, we can set up and solve the following integral:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 Ax^2y^2\,dx\,dy = \frac{1}{9}\cdot A,$$

hence, we can see that $\boxed{A = 9}$.

(b) Since the joint density can only be nonzero when X is between 0 and 1, we know that outside of this interval, the marginal density of X must also be zero. Inside this interval, we can find the marginal density of X by integrating the joint density with respect to Y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 9x^2y^2\,dy = 3x^2.$$

Thus, we can see that the marginal density of X is given by

$$f_X(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Verifying our answer, one can see that this function is both nonnegative and integrates to 1.

(c) There are two ways of approaching this. The first way is to use the fact that

$$\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

In order to apply this formula, we need to first find these values. We first find that

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 9x^3y^3\,dx\,dy = \frac{9}{16}.$$

Moreover, we can compute that

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf_X(x)\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}.$$

Finally, in order to compute $\mathbb{E}[Y]$, we first find the marginal density of Y. We can do this in a similar fashion to the previous part by integrating the joint density with respect to X. Since the joint density is zero when Y is not between 0 and 1, we know that the marginal density of Y must also be zero outside of this interval. When Y is inside this interval, we have that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 9x^2y^2\,dx = 3y^2,$$

hence the full marginal density is

$$f_Y(y) = \begin{cases} 3y^2 & \text{if } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

This then allows to compute

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 3y^3\, dy = \frac{3}{4},$$

hence

$$\text{cov}(X,Y) = \frac{9}{16} - \frac{3}{4} \cdot \frac{3}{4} = \boxed{0}.$$

The second way of approaching this problem is to first compute the marginal density of Y. Upon doing so, one can check that for any pair of values x, y, we have that

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y),$$

meaning that X and Y are independent random variables. Thus, since the covariance between two independent random variables is always zero, we can conclude the desired result and finish.

3 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

- What's the cumulative distribution function of X?
- What's the cumulative distribution function of Y?
- What's the probability density function of X?
- What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max\{X,Y\}$?

(d) What's the cumulative distribution function of $V = \min\{X,Y\}$?

Solution:

(a)
- To get the cumulative distribution function of X, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X \leq x] = \frac{\pi x^2}{\pi} = x^2, \qquad x \in [0,1].$$

- Using the same approach as the previous part:

$$\mathbb{P}[Y \leq y] = \frac{\pi y^2}{\pi \cdot 4} = \frac{y^2}{4}, \qquad y \in [0,2].$$

- We'll take the derivative of the CDF to get the following:

$$f_X(x) = \frac{\mathrm{d}\mathbb{P}[X \leq x]}{\mathrm{d}x} = 2x, \qquad x \in [0,1].$$

- Using the same approach as the previous part:

$$f_Y(y) = \frac{\mathrm{d}\mathbb{P}[Y \leq y]}{\mathrm{d}y} = \frac{y}{2}, \qquad y \in [0,2].$$

(b) We'll condition on Alex's outcome and then integrate over all the possibilities to get the marginal $\mathbb{P}(X \leq Y)$ as following:

$$\begin{aligned}\mathbb{P}[X \leq Y] &= \int_0^2 \mathbb{P}[X \leq Y \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_0^1 y^2 \times \frac{y}{2}\,\mathrm{d}y + \int_1^2 1 \times \frac{y}{2}\,\mathrm{d}y \\ &= \frac{1}{8} + \frac{3}{4} = \frac{7}{8}.\end{aligned}$$

Note the range within which $\mathbb{P}[X \leq Y] = 1$. This allowed us to separate the integral to simplify our solution. Using this, we can get $\mathbb{P}[Y \leq X]$ by the following:

$$\mathbb{P}[Y \leq X] = 1 - \mathbb{P}[X \leq Y] = \frac{1}{8}$$

A similar approach to the integral above could be used to verify this result:

$$\mathbb{P}[Y \leq X] = \int_0^1 \mathbb{P}[Y \leq X \mid X = x] f_X(x)\,\mathrm{d}x = \int_0^1 \frac{x^2}{4} 2x\,\mathrm{d}x = \frac{1}{2}\int_0^1 x^3\,\mathrm{d}x = \frac{1}{8}.$$

(c) Getting the CDF of U relies on the insight that for the maximum of two random variables to be smaller than a value, they both need to be smaller than that value. Using this we can get the following result for $u \in [0,1]$:

$$\mathbb{P}[U \leq u] = \mathbb{P}[X \leq u]\mathbb{P}[Y \leq u] = \left(u^2\right)\left(\frac{u^2}{4}\right) = \frac{u^4}{4}.$$

For $u \in [1,2]$ we have $\mathbb{P}[X \leq u] = 1$; this makes

$$\mathbb{P}[U \leq u] = \mathbb{P}[Y \leq u] = \frac{u^2}{4}.$$

For $u > 2$ we have $\mathbb{P}[U \leq u] = 1$ since CDFs of both X and Y are 1 in this range.

(d) Getting the CDF of V relies on a similar insight that for the minimum of two random variables to be greater than a value, they both need to be greater than that value. Taking the complement of this will give us the CDF of V. This allows us to get the following result. For $v \in [0,1]$:

$$\begin{aligned}\mathbb{P}[V \leq v] &= 1 - \mathbb{P}[V \geq v] = 1 - \mathbb{P}[X \geq v]\mathbb{P}[Y \geq v] = 1 - \big(1 - \mathbb{P}[X \leq v]\big)\big(1 - \mathbb{P}[Y \leq v]\big) \\ &= 1 - \left(1 - v^2\right)\left(1 - \frac{v^2}{4}\right) = \frac{5v^2}{4} - \frac{v^4}{4}.\end{aligned}$$

For $v > 1$, we get $\mathbb{P}[X > v] = 0$, making $\mathbb{P}[V \leq v] = 1$.