1   Box of Marbles

Note 14

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly, and pick a marble what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

Solution:

(a) Let B be the event that the picked marble is blue, R be the event that it is red, A_1 be the event that the marble is picked from box 1, and A_2 be the event that the marble is picked from box 2. Therefore we want to calculate $\mathbb{P}[B]$. By total probability,

$$\mathbb{P}[B] = \mathbb{P}[B \mid A_1] P[A_1] + \mathbb{P}[B \mid A_2] \mathbb{P}[A_2] = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3.$$

(b) In this part, we want to find $\mathbb{P}[A_1 \mid B]$. By Bayes rule,

$$\mathbb{P}[A_1 \mid B] = \frac{\mathbb{P}[B \mid A_1] \mathbb{P}[A_1]}{\mathbb{P}[B \mid A_1] \mathbb{P}[A_1] + \mathbb{P}[B \mid A_2] \mathbb{P}[A_2]} = \frac{0.1 \times 0.5}{0.5 \times 0.1 + 0.5 \times 0.5} = \frac{1}{6}.$$

(c) Let B_1 be the event that first marble is blue, R_1 be the event that the first marble is red, and B_2 be the event that second marble is blue without looking at the color of first marble. We want to find $\mathbb{P}[B_2]$. By total probability,

$$\mathbb{P}[B_2] = \mathbb{P}[B_2 \mid B_1] \mathbb{P}[B_1] + \mathbb{P}[B_2 \mid R_1] \mathbb{P}[R_1] = \frac{99}{999} \times 0.1 + \frac{100}{999} \times 0.9 = 0.1.$$

More generally, one can see that the probability that the n-th marble picked from box 1 is blue with probability 0.1. This is clear by symmetry: all the permutations of the 1000 marbles have the same probability, so the probability that the n-th marble is blue is the same as the probability that the first marble is blue.

2 Poisoned Smarties

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 50% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 40% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 10%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through her investigations, the inspector found that 2 Smarties out of every 100 at Kelly's factory was poisonous. At See's factory, 5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.1.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

Solution:

(a) Let S be the event that a smarty is safe to eat. Let BK be the event that a smarty is from Burr Kelly's factory. Let YS be the event that a smarty is from Yousef See's factory. Finally, let SF be the event that a smarty is from Stan Furd's factory.

By total probability, we have

$$\mathbb{P}[S] = \mathbb{P}[BK]\mathbb{P}[S \mid BK] + \mathbb{P}[YS]\mathbb{P}[S \mid YS] + \mathbb{P}[SF]\mathbb{P}[S \mid SF]$$
$$= \frac{1}{2} \cdot \frac{49}{50} + \frac{2}{5} \cdot \frac{19}{20} + \frac{1}{10} \cdot \frac{9}{10}$$
$$= \frac{49}{100} + \frac{38}{100} + \frac{9}{100}$$
$$= \frac{96}{100} = \frac{24}{25} = 0.96$$

Therefore the probability that a Smarty is safe to eat is 0.96.

(b) Let P be the event that a smarty is poisonous.

$$\mathbb{P}[P \mid \overline{BK}] = \frac{\mathbb{P}[\overline{BK} \cap P]}{\mathbb{P}[\overline{BK}]}$$

Since BK, YS, SF are a partition of the entire sample space, we know that if BK did not occur, then either YS occurred, or SF occurred:

$$= \frac{\mathbb{P}[YS \cap P]}{\mathbb{P}[\overline{BK}]} + \frac{\mathbb{P}[SF \cap P]}{\mathbb{P}[\overline{BK}]}$$

$$= \frac{\mathbb{P}[P \mid YS]\mathbb{P}[YS]}{1 - \mathbb{P}[BK]} + \frac{\mathbb{P}[P \mid SF]\mathbb{P}[SF]}{1 - \mathbb{P}[BK]}$$

$$= \frac{\frac{1}{20} \cdot \frac{2}{5}}{\frac{1}{2}} + \frac{\frac{1}{10} \cdot \frac{1}{10}}{\frac{1}{2}} = 2 \cdot \frac{2}{100} + 2 \cdot \frac{1}{100}$$

$$= \frac{6}{100} = \frac{3}{50} = 0.06$$

(c) From Bayes' Rule, we know that:

$$\mathbb{P}[SF \mid P] = \frac{\mathbb{P}[P \mid SF]\mathbb{P}[SF]}{\mathbb{P}[P]}.$$

In part (a), we calculated the probability that any random Smarty was safe to eat; here, notice that $\mathbb{P}[P] = 1 - \mathbb{P}[S]$. This means we have

$$\mathbb{P}[SF \mid P] = \frac{\mathbb{P}[P \mid SF]\mathbb{P}[SF]}{1 - \mathbb{P}[S]}$$

$$= \frac{\frac{1}{10} \cdot \frac{1}{10}}{1 - \frac{24}{25}} = \frac{\frac{1}{100}}{\frac{1}{25}}$$

$$= \frac{25}{100} = \frac{1}{4} = 0.25$$

3 Pairwise Independence

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

Solution:

(a) We have that $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \frac{1}{6}$, since we have a $\frac{1}{6}$ probability of getting a particular number on a fair die.

Since there are 6 ways in which the two dice can sum to 7 (i.e. $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$), we have $\mathbb{P}[A_3] = \frac{1}{6}$ as well.

(b) We want to determine whether $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\mathbb{P}[A_2]$. We already found the probabilities of A_1 and A_2 from part (a), so let's look at $\mathbb{P}[A_1 \cap A_2]$. There's only one possible outcome where the first die is a 1 and the second die is a 6, so this gives a probability of $\mathbb{P}[A_1 \cap A_2] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\mathbb{P}[A_2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_2]$, these two events are independent.

(c) We want to determine whether $\mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_2]\mathbb{P}[A_3]$. We already found the probabilities of A_2 and A_3 from part (a), so let's look at $\mathbb{P}[A_2 \cap A_3]$. These two events both occur if the second die lands on a 6, and the two dice sum to 7. There's only one way that this can happen, i.e. the first die must be a 1, so the intersection has probability $\mathbb{P}[A_2 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_2]\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_2 \cap A_3]$, these two events are independent.

(d) To see whether the three events are pairwise independent, we need to ensure that all pairs of events are independent. We've already checked that A_1 and A_2 are independent, and that A_2 and A_3 are independent, so it suffices to check whether A_1 and A_3 are independent.

Similar to the previous two parts, the intersection $A_1 \cap A_3$ means that the first die must land on a 1, and the two dice sum to 7. There's only one way for this to happen, i.e. the second die must land on a 6, so the probability is $\mathbb{P}[A_1 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_3]$, these two events are also independent. Since we've now shown that all possible pairs of events are independent, A_1, A_2, and A_3 are indeed pairwise independent.

(e) Mutual independence requires the additional constraint that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3]$. We've found the individual probabilities of these events in part (a), so we only need to compute $\mathbb{P}[A_1 \cap A_2 \cap A_3]$.

Here, we must have that the first die lands on 1, the second die lands on 6, and the sum of the two dice is equal to 7. There's only one way for this to happen, i.e. the first die is a 1 and the second die is a 6, so the probability of the intersection of all three events is $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{36}$.

However, since $\mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216} \neq \frac{1}{36} = \mathbb{P}[A_1 \cap A_2 \cap A_3]$, these three events are not mutually independent.