CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 8B

1 Aces

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:

(a) Inclusion-Exclusion Principle: $\frac{4}{52}+\frac{26}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$.

(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52}+\frac{13}{52}-2\cdot\frac{1}{52}=\frac{15}{52}$.

(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.

(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$.

(e) Complement to getting no aces: $\mathbb{P}[\text{at least one ace}]=1-\mathbb{P}[\text{zero aces}]=1-\frac{\binom{48}{5}}{\binom{52}{5}}$.

(f) Complement to getting no aces and no hearts: $\mathbb{P}[\text{at least one ace OR at least one heart}]=1-\mathbb{P}[\text{zero aces AND zero hearts}]=1-\frac{\binom{36}{5}}{\binom{52}{5}}$. This is because $52-13-3=36$, where 13 is the number of hearts and 3 is the number of non-heart aces.

2 Box of Marbles

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly, and pick a marble what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

Solution:

(a) Let B be the event that the picked marble is blue, R be the event that it is red, A_1 be the event that the marble is picked from box 1, and A_2 be the event that the marble is picked from box 2. Therefore we want to calculate $\mathbb{P}(B)$. By total probability,

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)P(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2) = 0.5 \times 0.1 + 0.5 \times 0.5 = 0.3.$$

(b) In this part, we want to find $\mathbb{P}(A_1 \mid B)$. By Bayes rule,

$$\mathbb{P}(A_1 \mid B) = \frac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_2)\mathbb{P}(A_2)} = \frac{0.1 \times 0.5}{0.5 \times 0.1 + 0.5 \times 0.5} = \frac{1}{6}.$$

(c) Let B_1 be the event that first marble is blue, R_1 be the event that the first marble is red, and B_2 be the event that second marble is blue without looking at the color of first marble. We want to find $\mathbb{P}(B_2)$. By total probability,

$$\mathbb{P}(B_2) = \mathbb{P}(B_2 \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(B_2 \mid R_1)\mathbb{P}(R_1) = \frac{99}{999} \times 0.1 + \frac{100}{999} \times 0.9 = 0.1.$$

More generally, one can see that the probability that the n-th marble picked from box 1 is blue with probability 0.1. This is clear by symmetry: all the permutations of the 1000 marbles have the same probability, so the probability that the n-th marble is blue is the same as the probability that the first marble is blue.

3 Mario's Coins

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

Solution:

(a) X_1 and X_2 are not independent. Intuitively, the fact that X_1 happened gives some information about the first coin that was chosen; this provides some information about the second coin that was chosen (since the first and second coins can't be the same coin), which directly affects whether X_2 happens or not.

To make this formal, we compute

$$\mathbb{P}(X_1) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}(X_2) = \mathbb{P}(X_1)$, so

$$\mathbb{P}(X_1)\mathbb{P}(X_2) = \frac{1}{4}.$$

But if we consider the probability that both X_1 and X_2 happen, we have

$$\begin{aligned}\mathbb{P}(X_1 \cap X_2) &= \frac{1}{6}\Big[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \\ &\qquad \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\Big] \\ &= \frac{22}{96} = \frac{11}{48}\end{aligned}$$

which is not equal to $1/4$, violating the definition of independence.

(b) Y_1 and Y_2 are not independent. Intuitively, the fact that Y_1 happens gives some information about the coin that was picked, which directly influences whether Y_2 happens or not.

To make this formal, we compute

$$\mathbb{P}(Y_1) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}(Y_2) = \mathbb{P}(Y_1)$, so

$$\mathbb{P}(Y_1)\mathbb{P}(Y_2) = \frac{1}{4}$$

But if we consider the probability that both Y_1 and Y_21 happen, we have

$$\mathbb{P}(Y_1 \cap Y_2) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24}$$

which is not equal to $1/4$, violating the definition of independence.

(c) Let A be the coin with bias $1/4$, B be the fair coin, and C be the coin with bias $3/4$. There are six orderings, each with probability $1/6$: ABC, ACB, BAC, BCA, CAB, and CBA. Thus

$$\begin{aligned}
&\mathbb{P}(\text{Third coin shows heads} \mid \text{First two coins show heads}) \\
&= \frac{\mathbb{P}(\text{All three coins show heads})}{\mathbb{P}(\text{First two coins show heads})} \\
&= \frac{(1/4)(1/2)(3/4)}{\sum_{\text{Orderings}} \mathbb{P}(\text{First two coins show heads} \mid \text{Ordering})\mathbb{P}(\text{Ordering})} \\
&= \frac{(1/4)(1/2)(3/4)}{(1/6)\sum_{\text{Orderings}} \mathbb{P}(\text{First two coins show heads} \mid \text{Ordering})} \\
&= \frac{(1/4)(1/2)(3/4)}{(1/6)((1/4)(1/2)+(1/4)(3/4)+(1/2)(1/4)+(1/2)(3/4)+(3/4)(1/4)+(3/4)(1/2))} \\
&= \frac{3/32}{11/48} = \frac{9}{22}.
\end{aligned}$$

4 Duelling Meteorologists

Tom is a meteorologist in New York. On days when it snows, Tom correctly predicts the snow 70% of the time. When it doesn't snow, he correctly predicts no snow 95% of the time. In New York, it snows on 10% of all days.

(a) If Tom says that it is going to snow, what is the probability it will actually snow?

(b) Let A be the event that, on a given day, Tom predicts the weather correctly. What is $\mathbb{P}(A)$?

(c) Tom's friend Jerry is a meteorologist in Alaska. Jerry claims that she is a better meteorologist than Tom even though her overall accuracy is lower. After looking at their records, you determine that Jerry is indeed better than Tom at predicting snow on snowy days and sun on sunny day. Give an instance of the situation described above. *Hint: what is the weather like in Alaska?*

Solution:

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned}
\mathbb{P}(S|T) &= \frac{\mathbb{P}(S \cap T)}{\mathbb{P}(T)} \\
&= \frac{\mathbb{P}(S) \cdot \mathbb{P}(T|S)}{\mathbb{P}(S \cap T) + \mathbb{P}(\overline{S} \cap T)} \\
&= \frac{\frac{1}{10} \times \frac{7}{10}}{\frac{1}{10} \times \frac{7}{10} + \frac{9}{10} \times \frac{5}{100}} = \frac{14}{23}
\end{aligned}$$

(b)

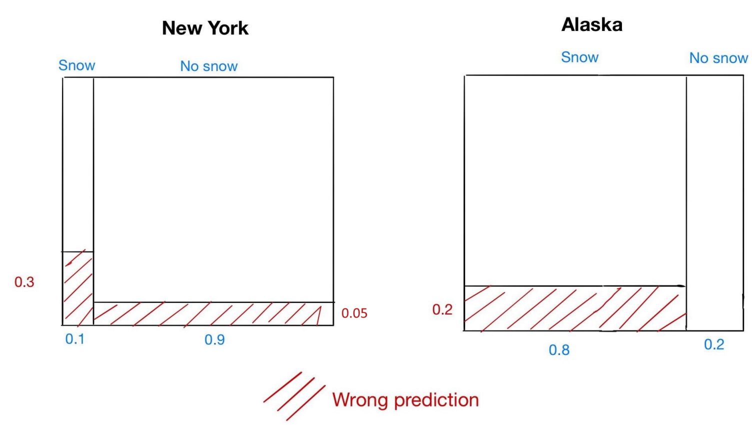
$$\mathbb{P}(A) = \mathbb{P}(S \cap T) + \mathbb{P}(\overline{S} \cap \overline{T})$$
$$= \frac{1}{10} \times \frac{7}{10} + \frac{9}{10} \times \frac{95}{100} = \frac{37}{40}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better meteorologist if the weather is different.

For example, let's assume that it snows 80% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Tom's even though she is better at predicting both categories! The following diagram gives an illustration of the situation. The intuition is that Jerry's error gets penalized more heavily than Tom because it snows more often in Alaska.



For more info on this kind of phenomena, check out Simpson's Paradox!