CS 70 Discrete Mathematics and Probability Theory
Fall 2026 Hug, Sabin

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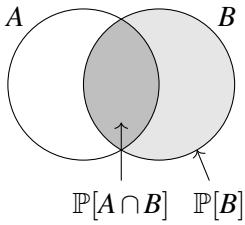
Conditional Probability Intro

Note 17
Note 18

Conditional Probability: Probability of event A, *given* that event B has happened (implying that $P[B] > 0$);

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Think of it like restricting our sample space:



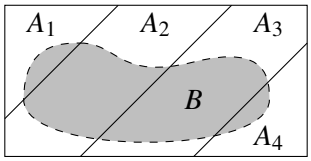
Bayes' Rule: A consequence of conditional probability - notice $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$, so

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[B]} = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}.$$

Total Probability Rule: If disjoint events $A_1, \ldots, A_n$ form a partition on the sample space Ω (meaning that each outcome in Ω belongs to exactly one of $A_1, \ldots, A_n$.), we then have

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\,\mathbb{P}[A_i].$$

Visually, we're splitting an event into partitions and looking at each intersection individually:



Most notably, we can always condition on the partition $\{A, \overline{A}\}$ to obtain the expression:

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap \overline{A}] = \mathbb{P}[B|A]\,\mathbb{P}[A] + \mathbb{P}[B|\overline{A}]\,\mathbb{P}[\overline{A}].$$

This formula is useful when we don't know the *total probability* $\mathbb{P}[B]$ (I don't care whether A happens or not) but know the conditioned probabilities $\mathbb{P}[B|A]$ and $\mathbb{P}[B|\overline{A}]$.

Independence: Two events are independent if the following (equivalent) conditions are satisfied. The second definition is probably more intuitive - B happening does not affect the probability of A happening.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$
$$\mathbb{P}[A \mid B] = \mathbb{P}[A]$$

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is a generalization of the definition of conditional probability:

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1].$$

Union Bound: The probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \leq \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$
$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

Inclusion-Exclusion: We also have PIE in probability! As with counting, the minimal example is:

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2].$$

And the general case is similar, albeit a bit notation-heavy:

$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] = \sum_i \mathbb{P}[A_i] - \sum_{i<j} \mathbb{P}[A_i \cap A_j] + \sum_{i<j<k} \mathbb{P}[A_i \cap A_j \cap A_k] - \cdots + (-1)^{n-1}\,\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n]$$

Notice that the Union Bound is just PIE without any correction terms for intersections of events. In practice, the Union Bound is one of the most beloved inequalities in probability because it is a quick, simple, and unconditional upper bound of the exact PIE expression.

1 Movie Archive

Note 17
Note 18

Sunny has a movie archive with a bunch of Blu-rays and DVDs. Help her figure out some probabilities about the types and genres of movies on her shelf. Sunny counts that out of her Blu-rays, she has 3 action movies and 5 comedies. Out of her DVDs, she has 4 action movies and 8 comedies.

(a) If she randomly selects a DVD from her archive, what is the probability that it is an action movie?

(b) Now Audrey comes to Sunny's house and randomly picks a movie to play. She doesn't know the difference between Blu-rays and DVDs, but the movie playing turns out to be a comedy. What is the probability that the movie she picked was a DVD?

(c) What is the probability of a random movie being either a DVD or a comedy?

(d) You have watched all of these movies, and know empirically that only 50% of action movies are good, while 80% of comedies are good. What is the chance of a randomly selected movie being good?

(e) Are the events of a movie being good and a movie being an action movie independent? Justify your answer.

Solution: Let B, D, A, C, and G be the events that a random movie is a Blu-ray, a DVD, an action movie, a comedy, and good, respectively. There are $3+5+4+8=20$ movies in total.

(a) There are 4 action DVDs out of 20 movies, so $\mathbb{P}[A \cap D] = \frac{4}{20}$. Since there are an additional 8 comedy DVDs, $\mathbb{P}[D] = \frac{4+8}{20} = \frac{12}{20}$. Thus, the probability of a random DVD being an action movie is $\mathbb{P}[A \mid D] = \frac{\mathbb{P}[A \cap D]}{\mathbb{P}[D]} = \frac{4/20}{12/20} = \frac{4}{12} = \frac{1}{3}$.

(b) By Bayes' Rule, $\mathbb{P}[D \mid C] = \frac{\mathbb{P}[C|D] \cdot \mathbb{P}[D]}{\mathbb{P}[C]}$. We have $\mathbb{P}[C \mid D] = \frac{8}{12}$, $\mathbb{P}[D] = \frac{12}{20}$, and $\mathbb{P}[C] = \frac{5+8}{20} = \frac{13}{20}$. Thus, $\mathbb{P}[D \mid C] = \frac{\frac{8}{12} \cdot \frac{12}{20}}{\frac{13}{20}} = \frac{8}{13}$.

(c) The principle of inclusion-exclusion tells us that $\mathbb{P}[D \cup C] = \mathbb{P}[D] + \mathbb{P}[C] - \mathbb{P}[D \cap C]$. This means $\mathbb{P}[D \cup C] = \frac{12}{20} + \frac{13}{20} - \frac{8}{20} = \frac{17}{20}$.

(d) From the Total Probability Rule, the probability of being good is equal to $\mathbb{P}[G] = \mathbb{P}[G \cap A] + \mathbb{P}[G \cap C]$. We have $\mathbb{P}[G \cap A] = \mathbb{P}[G \mid A] \cdot \mathbb{P}[A] = \frac{1}{2} \cdot \frac{7}{20} = \frac{7}{40}$ and $\mathbb{P}[G \cap C] = \mathbb{P}[G \mid C] \cdot \mathbb{P}[C] = \frac{4}{5} \cdot \frac{13}{20} = \frac{13}{25}$. Thus, $\mathbb{P}[G] = \frac{7}{40} + \frac{13}{25} = \frac{139}{200}$.

(e) We have previously found that $\mathbb{P}[A] = \frac{7}{20}$, and $\mathbb{P}[G] = \frac{139}{200}$. Now, $\mathbb{P}[A \cap G] = \frac{7}{40}$, making $\mathbb{P}[A] \cdot \mathbb{P}[G] = \frac{7}{20} \cdot \frac{139}{200} = \frac{973}{4000}$. Since $\mathbb{P}[A \cap G] \neq \mathbb{P}[A] \cdot \mathbb{P}[G]$, the events are not independent.

2 Pairwise Independence

Note 18

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

Solution:

(a) We have that $\mathbb{P}[A_1] = \mathbb{P}[A_2] = \frac{1}{6}$, since we have a $\frac{1}{6}$ probability of getting a particular number on a fair die.

Since there are 36 ways to roll two fair six-sided dice overall, and 6 of these ways to have the two dice sum to 7 (i.e. $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$), we have $\mathbb{P}[A_3] = \frac{6}{36} = \frac{1}{6}$ as well.

(b) We want to determine whether $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]$. We already found the probabilities of A_1 and A_2 from part (a), so let's look at $\mathbb{P}[A_1 \cap A_2]$. There's only one possible outcome where the first die is a 1 and the second die is a 6, so this gives a probability of $\mathbb{P}[A_1 \cap A_2] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_2] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_2]$, these two events are independent.

(c) We want to determine whether $\mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We already found the probabilities of A_2 and A_3 from part (a), so let's look at $\mathbb{P}[A_2 \cap A_3]$. These two events both occur if the second die lands on a 6, and the two dice sum to 7. There's only one way that this can happen, i.e. the first die must be a 1, so the intersection has probability $\mathbb{P}[A_2 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_2 \cap A_3]$, these two events are independent.

(d) To see whether the three events are pairwise independent, we need to ensure that all pairs of events are independent. We've already checked that A_1 and A_2 are independent, and that A_2 and A_3 are independent, so it suffices to check whether A_1 and A_3 are independent.

Similar to the previous two parts, the intersection $A_1 \cap A_3$ means that the first die must land on a 1, and the two dice sum to 7. There's only one way for this to happen, i.e. the second die must land on a 6, so the probability is $\mathbb{P}[A_1 \cap A_3] = \frac{1}{36}$.

Since $\mathbb{P}[A_1]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = \mathbb{P}[A_1 \cap A_3]$, these two events are also independent. Since we've now shown that all possible pairs of events are independent, A_1, A_2, and A_3 are indeed pairwise independent.

(e) Mutual independence requires the additional constraint that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$. We've found the individual probabilities of these events in part (a), so we only need to compute $\mathbb{P}[A_1 \cap A_2 \cap A_3]$.

Here, we must have that the first die lands on 1, the second die lands on 6, and the sum of the two dice is equal to 7. There's only one way for this to happen, i.e. the first die is a 1 and the second die is a 6, so the probability of the intersection of all three events is $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{36}$.

However, since $\mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3] = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216} \neq \frac{1}{36} = \mathbb{P}[A_1 \cap A_2 \cap A_3]$, these three events are not mutually independent.

3 Bag of Coins

Note 17
Note 18

Your friend Forrest has a bag of n coins. You know that k of those are biased with probability p (i.e. these coins have probability p of being heads), while the remaining are fair. Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag without replacement, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

Solution:

(a) The probability of picking F for the first coin is $(n-k)/n$. The probability of picking F for the second coin, after picking one fair coin already is $(n-k-1)/(n-1)$. The probability of picking B for the third coin is $k/(n-2)$. Thus, the probability of picking the exact sequence FFB is

$$\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

(b) We can use the Total Probability Rule, considering the possible outcomes for the first and second coins:

$$\{FF, FB, BF, BB\}.$$

Let T be the event that the third coin is biased. Then,

$$\mathbb{P}[T] = \mathbb{P}[T \mid FF]\,\mathbb{P}[FF] + \mathbb{P}[T \mid FB]\,\mathbb{P}[FB] + \mathbb{P}[T \mid BF]\,\mathbb{P}[BF] + \mathbb{P}[T \mid BB]\,\mathbb{P}[BB].$$

We know that there are k biased coins and $n-k$ fair coins. Thus,

$$\begin{aligned}\mathbb{P}[T] = &\frac{k}{n-2} \cdot \frac{(n-k)(n-k-1)}{n(n-1)} \\ &+ \frac{k-1}{n-2} \cdot \frac{(n-k)k}{n(n-1)} \\ &+ \frac{k-1}{n-2} \cdot \frac{k(n-k)}{n(n-1)} \\ &+ \frac{k-2}{n-2} \cdot \frac{k(k-1)}{n(n-1)}.\end{aligned}$$

Simplifying,

$$\begin{aligned}\mathbb{P}[T] &= \frac{k(n-k)(n-k-1) + 2k(k-1)(n-k) + k(k-1)(k-2)}{n(n-1)(n-2)} \\ &= \frac{k}{n}.\end{aligned}$$

A simpler approach is to use the notion of symmetry. We can envision this by laying out all the coins in a line. If you pick the third coin in the line, the probability that coin is biased is the same as the probability that the first coin in the line is biased, or second, or tenth, etc. If we have no information about any other coins, the probability that any single coin is biased is still k/n.

(c) Note that the probability of picking any sequence of two fair coins and a biased coin is the same, since each is a product with the same factors in a different order. It is in fact the probability from part (a). We need to multiply by the number of arrangements of biased and fair coins, however. So, the probability of picking any sequence with two fair coins is

$$\binom{3}{1} \frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

We additionally need to consider the probability of getting 3 fair coins.

$$\frac{(n-k)(n-k-1)(n-k-2)}{n(n-1)(n-2)}$$

These two cases are disjoint, so we sum them to get our answer:

$$\binom{3}{1} \frac{(n-k)(n-k-1)k}{n(n-1)(n-2)} + \frac{(n-k)(n-k-1)(n-k-2)}{n(n-1)(n-2)}$$

(d) We can apply Bayes' Rule. Let H denote the event that Forrest sees heads.

$$\mathbb{P}[B \mid H] = \frac{\mathbb{P}[H \mid B]\,\mathbb{P}[B]}{\mathbb{P}[H]}$$

Note that $\mathbb{P}[H \mid B] = p$, and that $\mathbb{P}[B] = k/n$ by the same symmetry argument as in part (b). We can now compute the denominator. Using the Total Probability Rule, we can expand $\mathbb{P}[H]$.

$$\begin{aligned}\mathbb{P}[H] &= \mathbb{P}[H \mid B]\,\mathbb{P}[B] + \mathbb{P}[H \mid F]\,\mathbb{P}[F] \\ &= p\frac{k}{n} + \frac{1}{2}\frac{n-k}{n} \\ &= \frac{2pk+n-k}{2n}\end{aligned}$$

We now combine both parts to get our answer:

$$\frac{p \cdot (k/n)}{(2pk+n-k)/(2n)} = \frac{2pk}{2pk+n-k}.$$

4 Balls and Bins, All Day Every Day

Note 17
Note 18

Suppose n balls are thrown into n labeled bins one at a time, each landing in a uniformly random bin independently of the others, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \leq k \leq n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

Solution:

(a) The probability that a particular ball lands in the first bin is $1/n$. We need exactly k balls to land in the first bin, which occurs with probability $(1/n)^k$, and we need exactly $n-k$ balls to land in a different bin, which occurs with probability $(1-1/n)^{n-k}$, and there are $\binom{n}{k}$ ways to choose which of the k balls land in first bin. Thus, the probability is $\binom{n}{k}(1/n)^k(1-1/n)^{n-k}$.

(b) This is the summation over $k = n/2, \ldots, n$ of the probabilities computed in the first part, i.e.,

$$\sum_{k=n/2}^{n} \binom{n}{k}(1/n)^k(1-1/n)^{n-k}.$$

(c) The event that some bin has at least half of the balls is the union of the events A_k, $k = 1, \ldots, n$, where A_k is the event that bin k has at least half of the balls. By the union bound, $\mathbb{P}(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} \mathbb{P}(A_k) = np$.

(d) The probability that the first bin has at least half of the balls is p; similarly, the probability that the second bin has at least half of the balls is also p. There is overlap between these two events, however: since there are only n balls, both events occur exactly when the first bin has half of the balls and the second bin has the other half. The probability of this event is $\binom{n}{n/2}n^{-n}$: there are n^n total possible

configurations for the n balls to land in the bins, but if we require exactly $n/2$ of the balls to land in the first bin and the remaining balls to land in the second bin, there are $\binom{n}{n/2}$ ways to choose which balls land in the first bin. By the principle of inclusion-exclusion, our desired probability is $p + p - \binom{n}{n/2}n^{-n} = 2p - \binom{n}{n/2}n^{-n}$.

(e) Condition on the number of balls in the bin. First we calculate the probability $\mathbb{P}(C_k)$, where C_k is the event that, in addition to the first ball you threw, an additional $k-1$ of the other $n-1$ balls landed in this bin, which by the reasoning in Part (a) has probability

$$\mathbb{P}(C_k) = \binom{n-1}{k-1}(1/n)^{k-1}(1-1/n)^{n-k} \quad .$$

If we let B be the event that we pick up the first ball we threw, then

$$\mathbb{P}(B \mid C_k) = 1/k$$

since we are equally likely to pick any of the k balls in the bin. Thus the overall probability we are looking for is, by an application of the Total Probability Rule,

$$\mathbb{P}(B) = \sum_{k=1}^{n} \mathbb{P}(C_k \cap B) = \sum_{k=1}^{n} \mathbb{P}(C_k)\,\mathbb{P}(B \mid C_k) = \sum_{k=1}^{n} \frac{1}{k}\binom{n-1}{k-1}\left(\frac{1}{n}\right)^{k-1}\left(1-\frac{1}{n}\right)^{n-k}.$$

5 Disease Diagnosis

Note 17

You have a high fever and go to the doctor to identify the cause. 1% of the people have Ebola, 10% of the people have the flu, and 89% have neither. Suppose that 100% of the people with Ebola have a high fever, 30% of the people with the flu have a high fever, and 2% of the people with neither have a high fever. Is it more likely that you have Ebola, the flu, or neither?

Solution:

Let E be the event that the patient has Ebola, F be the event that the patient has flu, and N be the event that the patient has neither. The event of having a high fever is H. We want to compare $\mathbb{P}[E \mid H]$, $\mathbb{P}[F \mid H]$, and $\mathbb{P}[N \mid H]$. We find each value using Bayes' Rule, where the shared denominator is $\mathbb{P}[H]$ expanded by the Total Probability Rule.

$$\begin{aligned}
\mathbb{P}[E \mid H] &= \frac{\mathbb{P}[H \mid E]\,\mathbb{P}[E]}{\mathbb{P}[H \mid E]\,\mathbb{P}[E] + \mathbb{P}[H \mid F]\,\mathbb{P}[F] + \mathbb{P}[H \mid N]\,\mathbb{P}[N]} \\
&= \frac{1 \times 0.01}{1 \times 0.01 + 0.3 \times 0.1 + 0.02 \times 0.89} \\
&\approx 0.173
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}[F \mid H] &= \frac{\mathbb{P}[H \mid F]\,\mathbb{P}[F]}{\mathbb{P}[H \mid E]\,\mathbb{P}[E] + \mathbb{P}[H \mid F]\,\mathbb{P}[F] + \mathbb{P}[H \mid N]\,\mathbb{P}[N]} \\
&= \frac{0.3 \times 0.1}{1 \times 0.01 + 0.3 \times 0.1 + 0.02 \times 0.89} \\
&\approx 0.519
\end{aligned}$$

$$\begin{aligned}\mathbb{P}[N \mid H] &= \frac{\mathbb{P}[H \mid N]\,\mathbb{P}[N]}{\mathbb{P}[H \mid E]\,\mathbb{P}[E] + \mathbb{P}[H \mid F]\,\mathbb{P}[F] + \mathbb{P}[H \mid N]\,\mathbb{P}[N]} \\ &= \frac{0.02 \times 0.89}{1 \times 0.01 + 0.3 \times 0.1 + 0.02 \times 0.89} \\ &\approx 0.308\end{aligned}$$

So flu is the most likely.

6 Boy or Girl Paradox

Note 17
Note 18

Mr. Smith has two children. Assume each child is independently a boy or a girl, each with probability $1/2$. You learn that at least one of his children is a boy.

(a) What is the probability that both children are boys?

(b) Now suppose you knock on Mr. Smith's front door and you are greeted by a boy who you correctly deduce to be Mr. Smith's older child. What is the probability that he has two boys? Compare your answer to the answer in part (a).

Now consider a family of n children, where each child is independently a boy or a girl, each with probability $1/2$.

(c) Suppose the family has n_g girls and n_b boys. Show that every girl has more brothers than every boy.

(d) Conditional on the youngest child being a girl, what is the probability of her having exactly k brothers? Similarly, conditional on the youngest child being a boy, what is the probability of him having exactly k brothers?

Solution:

(a) Let B_1 be the event that the first child is a boy, and B_2 be the event that the second child is a boy. We are asked to find $\mathbb{P}[(B_1 \cap B_2) \mid (B_1 \cup B_2)]$:

$$\begin{aligned}\mathbb{P}[(B_1 \cap B_2) \mid (B_1 \cup B_2)] &= \frac{\mathbb{P}[(B_1 \cap B_2) \cap (B_1 \cup B_2)]}{\mathbb{P}[B_1 \cup B_2]} \\ &= \frac{\mathbb{P}[B_1 \cap B_2]}{\mathbb{P}[B_1] + \mathbb{P}[B_2] - \mathbb{P}[B_1 \cap B_2]} \\ &= \frac{(1/2)(1/2)}{1/2 + 1/2 - 1/4} = \frac{1/4}{3/4} \\ &= \frac{1}{3}\end{aligned}$$

Note: It is tempting to think that because the children's genders are independent, the probability of the second child being a boy given that the first is a boy is simply $1/2$. While this is true, when we write it out in terms of events, we can see that this is not the quantity that we want. See part (b) for more details.

(b) In this part, we want to find $\mathbb{P}[(B_1 \cap B_2) \mid B_1]$:

$$\begin{aligned}\mathbb{P}[(B_1 \cap B_2) \mid B_1] &= \frac{\mathbb{P}[(B_1 \cap B_2) \cap B_1]}{\mathbb{P}[B_1]} \\ &= \frac{\mathbb{P}[B_1 \cap B_2]}{\mathbb{P}[B_1]} \\ &= \frac{1/4}{1/2} = \frac{1}{2}\end{aligned}$$

Note the distinction between this part and part (a), and that a common mistake in determining the answer to part (a) is solving part (b) instead.

(c) Every girl clearly has n_b brothers. However, a boy cannot be his own brother, so every boy has exactly $n_b - 1 < n_b$ brothers.

(d) Since children's genders come independently at probability $1/2$, the probability of the youngest child, regardless of their gender, having exactly k brothers is $\binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1}$. That is, the probability of any child having a certain number of brothers is independent of their gender. However, within a family girls will always have more brothers than boys. This is an instance of Simpson's Paradox.