CS 70 Discrete Mathematics and Probability Theory
Fall 2026 Hug, Sabin

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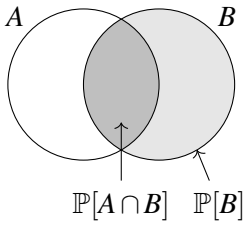
Conditional Probability Intro

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Conditional Probability: Probability of event A, *given* that event B has happened (implying that $P[B] > 0$);

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Think of it like restricting our sample space:



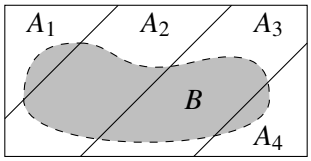
Bayes' Rule: A consequence of conditional probability - notice $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B]\,\mathbb{P}[B] = \mathbb{P}[B \mid A]\,\mathbb{P}[A]$, so

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \cap A]}{\mathbb{P}[B]} = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}.$$

Total Probability Rule: If disjoint events $A_1, \ldots, A_n$ form a partition on the sample space Ω (meaning that each outcome in Ω belongs to exactly one of $A_1, \ldots, A_n$.), we then have

$$\mathbb{P}[B] = \sum_{i=1}^{n} \mathbb{P}[B \cap A_i] = \sum_{i=1}^{n} \mathbb{P}[B \mid A_i]\,\mathbb{P}[A_i].$$

Visually, we're splitting an event into partitions and looking at each intersection individually:



Most notably, we can always condition on the partition $\{A, \overline{A}\}$ to obtain the expression:

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap \overline{A}] = \mathbb{P}[B|A]\,\mathbb{P}[A] + \mathbb{P}[B|\overline{A}]\,\mathbb{P}[\overline{A}].$$

This formula is useful when we don't know the *total probability* $\mathbb{P}[B]$ (I don't care whether A happens or not) but know the conditioned probabilities $\mathbb{P}[B|A]$ and $\mathbb{P}[B|\overline{A}]$.

Independence: Two events are independent if the following (equivalent) conditions are satisfied. The second definition is probably more intuitive - B happening does not affect the probability of A happening.

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$$
$$\mathbb{P}[A \mid B] = \mathbb{P}[A]$$

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is a generalization of the definition of conditional probability:

$$\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1].$$

Union Bound: The probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \leq \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$
$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

Inclusion-Exclusion: We also have PIE in probability! As with counting, the minimal example is:

$$\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2].$$

And the general case is similar, albeit a bit notation-heavy:

$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] = \sum_i \mathbb{P}[A_i] - \sum_{i<j} \mathbb{P}[A_i \cap A_j] + \sum_{i<j<k} \mathbb{P}[A_i \cap A_j \cap A_k] - \cdots + (-1)^{n-1}\,\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n]$$

Notice that the Union Bound is just PIE without any correction terms for intersections of events. In practice, the Union Bound is one of the most beloved inequalities in probability because it is a quick, simple, and unconditional upper bound of the exact PIE expression.

1 Movie Archive

Note 17
Note 18

Sunny has a movie archive with a bunch of Blu-rays and DVDs. Help her figure out some probabilities about the types and genres of movies on her shelf. Sunny counts that out of her Blu-rays, she has 3 action movies and 5 comedies. Out of her DVDs, she has 4 action movies and 8 comedies.

(a) If she randomly selects a DVD from her archive, what is the probability that it is an action movie?

(b) Now Audrey comes to Sunny's house and randomly picks a movie to play. She doesn't know the difference between Blu-rays and DVDs, but the movie playing turns out to be a comedy. What is the probability that the movie she picked was a DVD?

(c) What is the probability of a random movie being either a DVD or a comedy?

(d) You have watched all of these movies, and know empirically that only 50% of action movies are good, while 80% of comedies are good. What is the chance of a randomly selected movie being good?

(e) Are the events of a movie being good and a movie being an action movie independent? Justify your answer.

2 Pairwise Independence

Note 18

Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\,\mathbb{P}[A_2]\,\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

3 Bag of Coins

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Your friend Forrest has a bag of n coins. You know that k of those are biased with probability p (i.e. these coins have probability p of being heads), while the remaining are fair. Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag without replacement, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

4 Balls and Bins, All Day Every Day

Note 17
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Suppose n balls are thrown into n labeled bins one at a time, each landing in a uniformly random bin independently of the others, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \leq k \leq n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

5 Disease Diagnosis

Note 17

You have a high fever and go to the doctor to identify the cause. 1% of the people have Ebola, 10% of the people have the flu, and 89% have neither. Suppose that 100% of the people with Ebola have a high fever, 30% of the people with the flu have a high fever, and 2% of the people with neither have a high fever. Is it more likely that you have Ebola, the flu, or neither?

6 Boy or Girl Paradox

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Mr. Smith has two children. Assume each child is independently a boy or a girl, each with probability $1/2$. You learn that at least one of his children is a boy.

(a) What is the probability that both children are boys?

(b) Now suppose you knock on Mr. Smith's front door and you are greeted by a boy who you correctly deduce to be Mr. Smith's older child. What is the probability that he has two boys? Compare your answer to the answer in part (a).

Now consider a family of n children, where each child is independently a boy or a girl, each with probability $1/2$.

(c) Suppose the family has n_g girls and n_b boys. Show that every girl has more brothers than every boy.

(d) Conditional on the youngest child being a girl, what is the probability of her having exactly k brothers? Similarly, conditional on the youngest child being a boy, what is the probability of him having exactly k brothers?