CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 9A

# 1 Probability Potpourri

Provide brief justification for each part.

(a) For two events $A$ and $B$ in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of $C$. Prove that $D$ is independent of $C$.

(c) If $A$ and $B$ are disjoint, does that imply they're independent?

**Solution:**

(a) Start with the right side:

$$\begin{aligned}\mathbb{P}(A) - \mathbb{P}(B) &= [\mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B)] - [\mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)] \\ &= \mathbb{P}(A \setminus B) - \mathbb{P}(B \setminus A) \\ &\leq \mathbb{P}(A \setminus B)\end{aligned}$$

(b) Using total probability rule:

$$\mathbb{P}(D) = \mathbb{P}(D \cap C) + \mathbb{P}(D \cap \overline{C}) = \mathbb{P}(D \mid C) \cdot \mathbb{P}(C) + \mathbb{P}(D \mid \overline{C}) \cdot \mathbb{P}(\overline{C})$$

But we know that $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, so this simplifies to

$$\mathbb{P}(D) = \mathbb{P}(D \mid C) \cdot [\mathbb{P}(C) + \mathbb{P}(\overline{C})] = \mathbb{P}(D \mid C) \cdot 1 = \mathbb{P}(D \mid C)$$

which defines independence.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let $A$ be the event that we roll a 1, and let $B$ be the event that we roll a 2. Certainly $A$ and $B$ are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when disjoint $A$ and $B$ are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

## 2 Symmetric Marbles

A bag contains 4 red marbles and 4 blue marbles. Leanne and Sylvia play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Leanne wins if there are more red than blue marbles, and Sylvia wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

(a) Let $A_1$ be the event that the first marble is red and let $A_2$ be the event that the second marble is red. Are $A_1$ and $A_2$ independent?

(b) What is the probability that Leanne wins the game?

(c) Given that Leanne wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Leanne wins if the third marble is red, and Sylvia wins if the third marble is blue.

(d) What is the probability that the third marble is red?

(e) Given that there are $k$ red marbles among the four drawn, where $0 \le k \le 4$, what is the probability that the third marble is red? Answer in terms of $k$.

(f) Given that the third marble is red, what is the probability that Leanne wins the game?

**Solution:**

(a) They are not independent; removing one red marble lowers the probability of the next marble being red.

(b) Let $p$ be the probability that Leanne wins. Since there are an equal number of red and blue marbles, by symmetry, the probability that Leanne wins and the probability that Sylvia wins is the same. Thus, the probability that there is a tie is $1 - p - p = 1 - 2p$.

We now compute the probability that there is a tie. For there to be a tie, two of the four marbles need to be red. There are $\binom{8}{4}$ ways to pick 4 marbles, and $\binom{4}{2}\binom{4}{2}$ to pick 2 red and blue marbles, respectively, giving a probability of $\frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \frac{18}{35}$

We conclude that $1 - 2p = \frac{18}{35}$. Solving for $p$ gives $p = \boxed{\frac{17}{70}}$.

(c) Let $A$ be the event that there are 3 red marbles drawn, and let $B$ be the even that there are 4 red marbles drawn. We wish to compute $P(B|(A \cup B)) = \frac{P(B \cap (A \cup B))}{P(A \cup B)} = \frac{P(B)}{P(A)+P(B)}$. Similar to the calculation in part (a), the probability that there are 3 red marbles drawn is $\frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} = \frac{16}{70}$, and the probability that there are 4 red marbles drawn is $\frac{\binom{4}{4}\binom{4}{0}}{\binom{8}{4}} = \frac{1}{70}$, giving a final answer of $\frac{\frac{1}{70}}{\frac{16}{70} + \frac{1}{70}} = \frac{1}{17}$.

(d) By symmetry, the probability that the third marble is red is the same as the probability that the first marble is red, or the same as any marble being red. One way to see this is to imagine drawing the four marbles in order, then moving the first marble drawn to the third position. This is another way to draw four marbles that yields the same distribution.

There are 8 red marbles, and 12 marbles in total. Thus, the probability that the third marble is red is $\frac{8}{12} = \boxed{\frac{2}{3}}$.

(e) We are given that there are $k$ red marbles among the 4 drawn. By symmetry, each marble has the same probability of being red, so the probability that the third marble is red is $\boxed{\frac{k}{4}}$.

(f) The only way for Leanne to lose the game is if all the other marbles are blue. The probability that the third marble is red and all the other marbles are blue is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{495}$, and the probability that the third marble is red is $\frac{8}{12} = \frac{2}{3}$ so the probability that Leanne loses given that the third marble is red is $\frac{\frac{8}{495}}{\frac{2}{3}} = \frac{4}{165}$, and the probability that Leanne wins given that the third marble is red is $\boxed{\frac{161}{165}}$.

## 3 Poisoned Smarties

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 45% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through her investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

**Solution:**

(a) Let $S$ be the event that a smarty is safe to eat.
Let $BK$ be the event that a smarty is from Burr Kelly's factory.

Let $YS$ be the event that a smarty is from Yousef See's factory.
Finally, let $SF$ be the event that a smarty is from Stan Furd's factory.

$$\begin{aligned}\mathbb{P}(S) &= \mathbb{P}(BK)P(S \mid BK) + \mathbb{P}(YS)\mathbb{P}(S \mid YS) + \mathbb{P}(SF)\mathbb{P}(S \mid SF) \\ &= (0.45)(0.99) + (0.35)(0.985) + (0.2)(0.98) = 0.98625.\end{aligned}$$

Therefore the probability that a Smarty is safe to eat is about 0.98625.

(b) Let $P$ be the event that a smarty is poisonous.

$$\begin{aligned}\mathbb{P}(P \mid \neg BK) &= \frac{\mathbb{P}(\neg BK \cap P)}{\mathbb{P}(\neg BK)} \\ &= \frac{\mathbb{P}(YS \cap P)}{\mathbb{P}(\neg BK)} + \frac{\mathbb{P}(SF \cap P)}{\mathbb{P}(\neg BK)} \quad [\because BK, YS, SF \text{ are mutually exclusive, collectively exhaustive}] \\ &= \frac{\mathbb{P}(YS)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid YS) + \frac{\mathbb{P}(SF)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid SF) \\ &= \frac{0.35}{0.55} \cdot 0.015 + \frac{0.2}{0.55} \cdot 0.02 = 0.0168.\end{aligned}$$

(c) From Bayes' Rule, we know that:

$$\mathbb{P}(SF \mid P) = \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{\mathbb{P}(P)}$$

In the first part we calculate the probability that any random Smarty was safe to eat. We can use that since $\mathbb{P}(P) = 1 - \mathbb{P}(S)$. Therefore the solution becomes:

$$\begin{aligned}\mathbb{P}(SF \mid P) &= \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{1 - \mathbb{P}(S)} \\ &= \frac{(0.02)(0.2)}{(1 - 0.98625)} = 0.29.\end{aligned}$$