1 Probability Potpourri

Provide brief justification for each part.

(a) For two events \( A \) and \( B \) in any probability space, show that \( P[A \setminus B] \geq P[A] - P[B] \).

(b) Suppose \( P[D \mid C] = P[D \mid \overline{C}] \), where \( \overline{C} \) is the complement of \( C \). Prove that \( D \) is independent of \( C \).

(c) If \( A \) and \( B \) are disjoint, does that imply they’re independent?

Solution:

(a) It can be helpful to first draw out a Venn diagram:

![Venn Diagram]

We can see here that \( P[A] = P[A \cap B] + P[A \setminus B] \), and that \( P[B] = P[A \cap B] + P[B \setminus A] \).

Looking at the RHS, we have

\[
P[A] - P[B] = (P[A \cap B] + P[A \setminus B]) - (P[A \cap B] + P[B \setminus A]) \\
= P[A \setminus B] - P[B \setminus A] \\
\leq P[A \setminus B]
\]

(b) Using the total probability rule, we have

\[
P[D] = P[D \cap C] + P[D \cap \overline{C}] = P[D \mid C] \cdot P[C] + P[D \mid \overline{C}] \cdot P[\overline{C}].
\]

But we know that \( P[D \mid C] = P[D \mid \overline{C}] \), so this simplifies to

\[
P[D] = P[D \mid C] \cdot (P[C] + P[\overline{C}]) = P[D \mid C] \cdot 1 = P[D \mid C],
\]

which defines independence.

(c) No; if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let \( A \) be the event that we roll a 1, and let \( B \) be the event that we roll a 2. Certainly \( A \) and \( B \) are disjoint, as \( P[A \cap B] = 0 \). But these events are not independent: \( P[B \mid A] = 0 \), but \( P[B] = 1/6 \).

Since disjoint events have \( P[A \cap B] = 0 \), we can see that the only time when disjoint \( A \) and \( B \) are independent is when either \( P[A] = 0 \) or \( P[B] = 0 \).
2 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You’re given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs (uniformly) without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

Solution:

(a) \[ P[\text{chocolate egg}] = \frac{20}{60} = \frac{1}{3}. \]

(b) Long calculation using Total Probability Rule: let \( C_i \) denote the event that the \( i \)th egg is chocolate, and \( P_i \) denote the event that the \( i \)th egg is plastic. We have

\[
P[C_2] = P[C_1 \cap C_2] + P[P_1 \cap C_2]
= P[C_1]P[C_2 | C_1] + P[P_1]P[C_2 | P_1]
= \frac{1}{3} \cdot \frac{19}{59} + \frac{2}{3} \cdot \frac{20}{59}
= \frac{1}{3}.
\]

Short calculation: By symmetry, this is the same probability as part (a), 1/3. This is because we don’t know what type of egg was picked on the first draw, so the distribution for the second egg is the same as that of the first. To see this rigorously observe that \( P[C_2 \cap P_1] = P[P_2 \cap C_1] \) and, thus:

\[
P[C_2] = P[C_2 \cap C_1] + P[C_2 \cap P_1]
= P[C_2 \cap C_1] + P[P_2 \cap C_1]
= P[C_1]
\]

(c) By symmetry, since we don’t know any information about the 2nd, 3rd, or 4th eggs, we have

\[ P[5\text{th egg }= \text{ plastic }| \text{ 1st egg }= \text{ plastic}] = P[2\text{nd egg }= \text{ plastic }| \text{ 1st egg }= \text{ plastic}] = \frac{39}{59}. \]

Rigorously, notice that \( P[C_5 \cap P_2 | P_1] = P[P_5 \cap C_2 | P_1] \) and therefore:

\[
P[P_5 | P_1] = P[P_5 \cap C_2 | P_1] + P[P_5 \cap P_2 | P_1]
= P[C_5 \cap P_2 | P_1] + P[P_5 \cap P_2 | P_1]
= P[P_5 | P_1]
\]

One could also brute force this with Total Probability Rule (like in the previous part), but the calculation is quite tedious.
3 Poisoned Smarties

Supposed there are 3 people who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the owners, produces considerably more Smarties than her competitors and has a commanding 45% of the market share. Yousef See, who inherited her riches, lags behind Burr and produces 35% of the world’s Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through her investigations, the inspector found that one Smarty out of every 100 at Kelly’s factory was poisonous. At See’s factory, 1.5% of Smarties produced were poisonous. And at Furd’s factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn’t come from Burr Kelly’s factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd’s Smarties Factory?

Solution:

(a) Let $S$ be the event that a smarty is safe to eat. Let $BK$ be the event that a smarty is from Burr Kelly’s factory. Let $YS$ be the event that a Smarty is from Yousef See’s factory. Finally, let $SF$ be the event that a smarty is from Stan Furd’s factory.

By total probability, we have


$$= (0.45)(0.99) + (0.35)(0.985) + (0.2)(0.98) = 0.98625.$$ 

Therefore the probability that a Smarty is safe to eat is about 0.98625.

(b) Let $P$ be the event that a smarty is poisonous.

$$P[P | BK] = \frac{P[BK \cap P]}{P[BK]}$$

Since $BK, YS, SF$ are a partition of the entire sample space, we know that if $BK$ did not occur, then either $YS$ occurred, or $SF$ occurred:

$$= \frac{P[YS \cap P] + P[SF \cap P]}{P[BK]}$$

$$= \frac{P[P | YS]P[YS]}{P[BK]} + \frac{P[P | SF]P[SF]}{1 - P[BK]}$$

$$= \frac{0.015 \cdot 0.35}{0.55} + \frac{0.02 \cdot 0.2}{0.55} = 0.0168.$$ 

(c) From Bayes’ Rule, we know that:

$$P[SF | P] = \frac{P[P | SF]P[SF]}{P[P]}.$$
In part (a), we calculated the probability that any random Smarty was safe to eat; here, notice that $\Pr[P] = 1 - \Pr[S]$. This means we have

\[
\Pr[SF \mid P] = \frac{\Pr[P \mid SF] \Pr[SF]}{1 - \Pr[S]}
\]

\[
= \frac{0.02 \cdot 0.2}{1 - 0.98625} = 0.29.
\]