CS 70 Discrete Mathematics and Probability Theory
Fall 2026 Hug, Sabin

# DIS 10

## Random Variables and Expectation Intro

Note 19
Note 20

**Random Variable**: A random variable $X$ is a function $X : \Omega \to \mathbb{R}$ that assigns a real number $X(\omega)$ to each outcome $\omega$. The function itself is not random; what is random is which outcome $\omega$ occurs. The event "$X = a$" is shorthand for $\{\omega \in \Omega : X(\omega) = a\}$.

**Distribution**: The distribution of $X$ is the collection $\{(a, \mathbb{P}[X = a]) : a \in \mathscr{A}\}$, where $\mathscr{A}$ is the set of values $X$ can take. In other words, it lists every value $X$ can take, along with the probability of each value. The events $X = a$ partition $\Omega$, so

$$\sum_{a \in \mathscr{A}} \mathbb{P}[X = a] = 1.$$

**Common Discrete Distributions**: Here, "success" means an event with probability $p$, and all trials are independent.

| **Distribution** | **Models** | $\mathbb{P}[X = k]$ | $\mathbb{E}[X]$ |
|---|---|---|---|
| $X \sim \text{Bernoulli}(p)$ | one trial; 1 if success | $p$ if $k = 1$, $1 - p$ if $k = 0$ | $p$ |
| $X \sim \text{Binomial}(n, p)$ | # successes in $n$ trials | $\binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, \ldots, n$ | $np$ |
| $X \sim \text{Uniform}\{1, \ldots, n\}$ | each value equally likely | $\frac{1}{n}$, $k = 1, \ldots, n$ | $\frac{n+1}{2}$ |
| $X \sim \text{Geometric}(p)$ | # trials until first success (inclusive) | $(1-p)^{k-1} p$, $k = 1, 2, \ldots$ | $\frac{1}{p}$ |
| $X \sim \text{Poisson}(\lambda)$ | # of rare events, average rate $\lambda$ | $\frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, \ldots$ | $\lambda$ |

For $X \sim \text{Geometric}(p)$, $\mathbb{P}[X > k] = (1-p)^k$ (the first $k$ trials all fail). This gives the *memoryless property*: $\mathbb{P}[X > n + m \mid X > m] = \mathbb{P}[X > n]$. If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent, then $X + Y \sim \text{Poisson}(\lambda + \mu)$.

**Expectation**: A weighted average of the values of $X$, weighted by their probabilities:

$$\mathbb{E}[X] = \sum_{a \in \mathscr{A}} a \cdot \mathbb{P}[X = a] = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}[\omega].$$

Note that $\mathbb{E}[X]$ need not be a value that $X$ can actually take (e.g., a die roll has $\mathbb{E}[X] = \frac{7}{2}$).

**Tail Sum Formula**: If $X$ takes values in $\{0, 1, 2, \ldots\}$, then

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i].$$

**Linearity of Expectation**: For *any* random variables $X, Y$ on the same probability space and constants $a, b$,

$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y].$$

This holds *whether or not* $X$ and $Y$ are independent. However, it does *not* say that $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ or $\mathbb{E}[\frac{1}{X}] = \frac{1}{\mathbb{E}[X]}$; these fail in general.

**Indicator Variables**: The indicator $I_A$ of an event $A$ is 1 if $A$ occurs and 0 otherwise, so $I_A \sim \text{Bernoulli}(\mathbb{P}[A])$ and $\mathbb{E}[I_A] = \mathbb{P}[A]$. To find the expectation of a random variable that *counts* something, write it as a sum of indicators $X = I_1 + \cdots + I_n$, where $I_i$ is the indicator of event $A_i$, and apply linearity:

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \cdots + \mathbb{E}[I_n] = \mathbb{P}[A_1] + \cdots + \mathbb{P}[A_n].$$

**Joint Distribution**: The joint distribution of $X$ and $Y$ is the collection of probabilities $\mathbb{P}[X = a, Y = b]$ for all $a \in \mathscr{A}$, $b \in \mathscr{B}$. The *marginal distributions* are found by summing out the other variable:

$$\mathbb{P}[X = a] = \sum_{b \in \mathscr{B}} \mathbb{P}[X = a, Y = b] \qquad \mathbb{P}[Y = b] = \sum_{a \in \mathscr{A}} \mathbb{P}[X = a, Y = b].$$

We can also condition one random variable on another:

$$\mathbb{P}[X = a \mid Y = b] = \frac{\mathbb{P}[X = a, Y = b]}{\mathbb{P}[Y = b]}.$$

**Independence of Random Variables**: $X$ and $Y$ are independent if the events $X = a$ and $Y = b$ are independent for *all* $a, b$:

$$\mathbb{P}[X = a, Y = b] = \mathbb{P}[X = a]\,\mathbb{P}[Y = b] \qquad \text{for all } a, b.$$

To show $X$ and $Y$ are *not* independent, it is enough to find a single pair $(a, b)$ where this fails. Mutual independence of more than two random variables is defined similarly. For example, indicators of independent coin tosses are mutually independent and identically distributed (i.i.d.).

# 1 Head Count

Note 19

Consider a coin with $\mathbb{P}[\text{Heads}] = 2/5$. Suppose you flip the coin 20 times, and define $X$ to be the number of heads.

(a) What is $\mathbb{P}[X = k]$, for some $0 \leq k \leq 20$? Express your answer in terms of $k$. (Do not just copy down a formula—re-derive it yourself!)

(b) What is the name of the distribution of $X$, and what are its parameters?

(c) What is $\mathbb{P}[12 \leq X \leq 14]$?

(d) What is $\mathbb{P}[X \geq 1]$? *Hint: You should be able to do this without a summation.*

(e) Now consider a second coin also with $\mathbb{P}[\text{Heads}] = 2/5$. Suppose you flip this second coin 30 times, and define $Y$ to be the number of heads. What is the distribution of the *total* number of heads among these two coins, i.e. what is the distribution of $X + Y$?

# 2 Head Count II

Note 19

Consider a coin with $\mathbb{P}[\text{Heads}] = 3/4$. Suppose you flip the coin until you see heads for the first time, and define $X$ to be the number of times you flipped the coin.

(a) What is $\mathbb{P}[X = k]$, for some $k \geq 1$? Express your answer in terms of $k$. (Do not just copy down a formula—re-derive it yourself!)

(b) What is the name of the distribution of $X$, and what are its parameters?

(c) What is $\mathbb{P}[X > k]$, for some $k \geq 0$? (You should not have any summations.)

(d) What is $\mathbb{P}[X < k]$, for some $k \geq 1$? (You should not have any summations.)

(e) What is $\mathbb{P}[X > k \mid X > m]$, for some $k \geq m \geq 0$? Show that your answer is equal to $\mathbb{P}[X > k - m]$. Why do we call this the memoryless property?

(f) Suppose $Y \sim \text{Geometric}(p)$ and $Z \sim \text{Geometric}(q)$ are independent. Find the distribution of $\min(Y, Z)$ and justify your answer.

*Hint:* consider flipping two coins (with $\mathbb{P}[\text{Heads}] = p$ and $\mathbb{P}[\text{Heads}] = q$ respectively) simultaneously.

# 3 Shuttles and Taxis at Airport

Note 20

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson distribution. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

(a) Write in terms of a Poisson distribution, the distribution of the number of taxis that arrive between 00:00 and 00:01:

(b) What is the distribution of the following:

(i) The number of taxis that arrive between times 00:00 and 00:20?

(ii) The number of shuttles that arrive between times 00:00 and 00:20?

(iii) The total number of vehicles (shuttles and taxis) that arrive between times 00:00 and 00:20?

(c) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

(d) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?

(e) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

# 4 Linearity

Note 20

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game $A$ 10 times and game $B$ 20 times. Each time you play game $A$, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game $B$ is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears? (*Hint*: Consider where the sequence "book" can appear in the string.)

(c) Sam repeatedly flips a coin with probability of Heads = 0.2 until he gets Heads. Then he rolls a fair 6-sided die with sides $\{1, 1, 1, 1, 2, 3\}$, and flips the coin that many more times. What is the expected number of coin flips Sam makes in total?

# 5 Throwing Frisbees

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Andy and $n-1$ other students play a frisbee-passing game, where $n \geq 3$. Andy makes the first throw. On each throw, the current holder chooses uniformly at random from the other $n-1$ people, using a fresh random choice. Nobody can throw to themselves. The game stops the first time the frisbee returns to Andy.

(a) Let $T$ be the total number of throws, including Andy's first throw and the final throw back to Andy. Identify the distribution of $T-1$ and find $\mathbb{E}[T]$.

(b) A student concludes from part (a) that everyone gets a turn on average. Find the expected number of students *other than Andy* who never receive the frisbee during the game. Also find the expected number of those students who do receive it.

Explain why the expected number of throws alone does not determine how many different people participate.

# 6 Family Planning

Note 20

Mr. and Mrs. Johnson decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let $G$ denote the number of girls that the Johnsons have. Let $C$ be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of $G$ and $C$. Fill in the table below.

| | $C=1$ | $C=2$ | $C=3$ |
|---|---|---|---|
| $G=0$ | | | |
| $G=1$ | | | |

(c) Use the joint distribution to compute the marginal distributions of $G$ and $C$ and confirm that the values are as you'd expect. Fill in the tables below.

| $\mathbb{P}[G=0]$ | |
|---|---|
| $\mathbb{P}[G=1]$ | |

| $\mathbb{P}[C=1]$ | $\mathbb{P}[C=2]$ | $\mathbb{P}[C=3]$ |
|---|---|---|
| | | |

(d) Are $G$ and $C$ independent?

(e) What is the expected number of girls the Johnsons will have? What is the expected number of children that the Johnsons will have?

# 7 Pairwise and Mutual Independence

Note 20

Let $X$ and $Y$ be independent Bernoulli random variables with parameter $p$, where $0 < p < 1$. Define

$$Z = \begin{cases} 1, & X \neq Y, \\ 0, & X = Y. \end{cases}$$

This operation is also called exclusive OR (XOR).

(a) Find the joint distribution of $X$ and $Z$, and the marginal distribution of $Z$.

(b) For which values of $p$ are $X$ and $Z$ independent? What about $Y$ and $Z$? Justify your answers.

(c) Set $p = 1/2$. Every pair among $X, Y, Z$ is independent. Are all three mutually independent? Prove your answer.

How can every pair be independent even though knowing any two variables determines the third?