1 Head Count II

Consider a coin with $\Pr[\text{Heads}] = \frac{3}{4}$. Suppose you flip the coin until you see heads for the first time, and define $X$ to be the number of times you flipped the coin.

(a) What is $\Pr[X = k]$, for some $k \geq 1$?

(b) Name the distribution of $X$ and what its parameters are.

(c) What is $\Pr[X \geq k]$, for some $k \geq 1$?

(d) What is $\Pr[X \leq k]$, for some $k \geq 1$?

(e) What is $\Pr[X \geq k \mid X \geq m]$, for some $k \geq m \geq 1$? How does this relate to $\Pr[X \geq k - (m - 1)]$?

Solution:

(a) If we flipped $k$ times, then we had $k - 1$ tails and 1 head, in that order, giving us

$$\Pr[X = k] = \frac{3}{4} \left(1 - \frac{3}{4}\right)^{k-1} = \frac{3}{4} \left(\frac{1}{4}\right)^{k-1}.$$ 

(b) $X \sim \text{Geometric}\left(\frac{3}{4}\right)$

(c) If we had to flip at least $k$ times before seeing our first heads, then our first $k - 1$ flips must have been tails, giving us

$$\Pr[X \geq k] = \left(1 - \frac{3}{4}\right)^{k-1} = \left(\frac{1}{4}\right)^{k-1}.$$ 

(d) Notice $\Pr[X \leq k] = 1 - \Pr[X > k] = 1 - \Pr[X \geq k + 1]$ since $X$ can only take on integer values. Along similar lines to the previous part, we then have

$$\Pr[X \leq k] = 1 - \Pr[X \geq k + 1] = 1 - \left(1 - \frac{3}{4}\right)^k = 1 - \left(\frac{1}{4}\right)^k.$$ 

(e) By part (c), we have

$$\Pr[X \geq k \mid X \geq m] = \frac{\Pr[X \geq k \cap X \geq m]}{\Pr[X \geq m]} = \frac{\Pr[X \geq k]}{\Pr[X \geq m]} = \left(\frac{1}{4}\right)^{k-m}.$$ 

However, note that this is exactly $\Pr[X \geq k - (m - 1)]$. The reason this makes sense is that if we want to compute the probability that the first heads occurs after $k$ flips, and we know that the first heads occurs after $m$ flips, then the first $m - 1$ flips are tails. Thus, by the independence of the coin flips, the first $m - 1$ flips don’t matter, and so we only need to compute the probability that the first heads occurs after $k - (m - 1)$ flips. This is called the memorylessness property of the geometric distribution.
2 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let $G$ denote the numbers of girls that the Browns have. Let $C$ be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of $G$ and $C$. Fill in the table below.

<table>
<thead>
<tr>
<th></th>
<th>$C = 1$</th>
<th>$C = 2$</th>
<th>$C = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Use the joint distribution to compute the marginal distributions of $G$ and $C$ and confirm that the values are as you’d expect. Fill in the tables below.

<table>
<thead>
<tr>
<th>$G = 0$</th>
<th>$P(C = 1)$</th>
<th>$P(C = 2)$</th>
<th>$P(C = 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = 1$</td>
<td>$P(G = 1)$</td>
<td>$P(G = 1)$</td>
<td>$P(G = 1)$</td>
</tr>
</tbody>
</table>

(d) Are $G$ and $C$ independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution:

(a) The sample space is the set of all possible sequences of children that the Browns can have: $\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:

- $P[g] = \frac{1}{2}$
- $P[bg] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
- $P[bbg] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
- $P[bbb] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

(b) | $C = 1$ | $C = 2$ | $C = 3$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = 0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G = 1$</td>
<td>$P[g] = 1/2$</td>
<td>$P[bg] = 1/4$</td>
</tr>
</tbody>
</table>

(c) Marginal distribution for $G$:

- $P[G = 0] = 0 + 0 + \frac{1}{8} = \frac{1}{8}$
- $P[G = 1] = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
Marginal distribution for $C$:

\[
\begin{align*}
\mathbb{P}[C = 1] &= 0 + \frac{1}{2} = \frac{1}{2} \\
\mathbb{P}[C = 2] &= 0 + \frac{1}{4} = \frac{1}{4} \\
\mathbb{P}[C = 3] &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}
\end{align*}
\]

(d) No, $G$ and $C$ are not independent. If two random variables are independent, then

\[
\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \mathbb{P}[Y = y].
\]

To show this dependence, consider an entry in the joint distribution table, such as $\mathbb{P}[G = 0, C = 3] = \frac{1}{8}$. This is not equal to $\mathbb{P}[G = 0] \mathbb{P}[C = 3] = (\frac{1}{8}) \cdot (\frac{1}{4}) = \frac{1}{32}$, so the random variables are not independent.

(e) We can apply the definition of expectation directly for this problem, since we’ve computed the marginal distribution for both random variables.

\[
\begin{align*}
\mathbb{E}[G] &= 0 \cdot \mathbb{P}[G = 0] + 1 \cdot \mathbb{P}[G = 1] = 1 \cdot \frac{7}{8} = \frac{7}{8} \\
\mathbb{E}[C] &= 1 \cdot \mathbb{P}[C = 1] + 2 \cdot \mathbb{P}[C = 2] + 3 \cdot \mathbb{P}[C = 3] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{7}{4}
\end{align*}
\]

3 Pullout Balls

Suppose you have a bag containing four balls numbered 1, 2, 3, 4.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

Solution:

(a) Let $X$ be the number that you record. Each ball is equally likely to be chosen, so

\[
\mathbb{E}[X] = \sum_{x} x \cdot \mathbb{P}[X = x] = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = 2.5
\]

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome $\omega$ for which $X(\omega) = 2.5$).

(b) Let $Y$ be the product of two numbers that you pull out. Then

\[
\mathbb{E}[Y] = \frac{1}{4} \binom{4}{2} (1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4) = \frac{2 + 3 + 4 + 6 + 8 + 12}{6} = \frac{35}{6}
\]
4 Double Counting

In this problem, we will show that linearity of expectation can be seen as a form of double counting.

Suppose Alice has two red and two blue marbles, where marbles of the same color are indistinguishable. She places the four marbles uniformly at random in a row. Let $X$ be the random variable representing the number of times that a red and blue marble are adjacent.

(a) Fill in the following table:

<table>
<thead>
<tr>
<th>Arrangement of marbles</th>
<th>1st and 2nd are RB or BR?</th>
<th>2nd and 3rd are RB or BR?</th>
<th>3rd and 4th are RB or BR?</th>
<th>Total number of RB or BR</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRBB</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>RBRB</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>BRRB</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>RBBR</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>BRBR</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>BBRR</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Without using linearity of expectation, compute $\mathbb{E}[X]$. (Hint: The table in part (a) will be helpful.)

(c) Now, let $I_1$, $I_2$, and $I_3$ be indicator variables, where $I_i$ is 1 if the $i$th and $i+1$th marble are an adjacent red-blue or blue-red pair, and 0 if not. Without using the table in part (a), compute $\mathbb{E}[I_1]$.

(d) Using linearity of expectation, compute $\mathbb{E}[X]$. 

(e) Argue why the different approaches in part (b) and part (d) should give the same answer. (Hint: Each of the expected values you computed corresponds to a different column of the table in part (a).)

**Solution:**

(a) We write down 1 if the RB or BR pair exists, and 0 otherwise.

<table>
<thead>
<tr>
<th>Arrangement of marbles</th>
<th>1st and 2nd are RB or BR?</th>
<th>2nd and 3rd are RB or BR?</th>
<th>3rd and 4th are RB or BR?</th>
<th>Total number of RB or BR</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRBB</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>RBRB</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>BRRB</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>RBBR</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>BRBR</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>BBRR</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) Each of the six marble arrangements are equally likely, so by the numbers in the last column of the table,

$$\mathbb{E}[X] = \frac{1}{6} \cdot (1 + 3 + 2 + 2 + 3 + 1) = 2.$$
(c) Since $I_1$ is an indicator variable, $\mathbb{E}[I_1]$ is equivalent to the probability that the 1st and 2nd marble are RB or BR. The probability that the second marble is different from the first is $\frac{2}{3}$, as given any choice of color for the first marble, there are 2 out of 3 choices for the second marble to differ in color. Thus, $\mathbb{E}[I_1] = \frac{2}{3}$.

(d) By symmetry, $\mathbb{E}[I_1] = \mathbb{E}[I_2] = \mathbb{E}[I_3] = \frac{2}{3}$. Thus,

$$\mathbb{E}[X] = \mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3] = 2.$$ 

(e) Note that $\mathbb{E}[I_i]$ is exactly equal to the number of 1s in the $i$th column (divided by 6). Thus, $\mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3]$ is exactly equal to the number of 1s in the first 3 columns (divided by 6). However, note that $\mathbb{E}[X]$ counts exactly the same thing; it counts the number of 1s in the first three columns first by adding them up by row, then adding the row counts together (then dividing everything by 6). Since $\mathbb{E}[I_1] + \mathbb{E}[I_2] + \mathbb{E}[I_3]$ and $\mathbb{E}[X]$ count the same thing, they must be equal.