# 1 Beast Arcade

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

(a) In the first game, Chandler hands you a crisp \$20 bill. Then, he flips a coin that shows heads with probability $p$ until a heads comes up. You receive an additional dollar for each flip. How much money will you get in expectation?

(b) In the next game, Karl rolls a fair 6-sided die. He then calculates $2^x$, where $x$ is the result of that die and hands you that much money. What is the expected amount of money you'll receive?

(c) For the last game, Jimmy makes your friend flip a fair coin $10,000$ times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with \$$1,000$ and says he will also pay you \$5 for each head that comes up. Let $X$ be a random variable representing the number of heads your friend flips. Use it to come up with an expression for $Y$, a random variable representing the total amount of money you'll receive.

(d) What is $E[Y]$? What about $P[Y = 26,000]$?

**Solution:**

(a) Let $X$ be a random variable representing the total number of flips needed for a heads. Notice that $X$ is geometric with parameter $p$, and thus its expected value is $\frac{1}{p}$. Then, we see that $Y$, the total amount of money we'll make, can be written as $20+X$, so $E[Y] = E[X]+E[20] = 20+\frac{1}{p}$.

(b) In game 2, we can use the formula $E[g(X)] = \sum_x g(x)p_X(x)$. In english, we'll sum how much money we'll get in each possible outcome multiplied by its likelihood. Since all 6 dice roll outcomes are equally likely, we know that $\forall x, p_X(x) = \frac{1}{6}$. This gives us $\sum_{i=1}^{6} \frac{1}{6} 2^i = 21$.

(c) We once again have a linear expression for $Y$ in terms of $X$. This means we can write $Y = 5X + 1000$.

(d) $E[Y] = E[5X + 1000] = E[5x] + E[1000] = 5E[X] + 1000 = 5 * 5000 + 1000 = 26000$. To calculate $P[Y = 26000]$, we note that you receive exactly \$26000 if and only if your friend flips exactly 5000 heads. This happens with probability $\binom{10000}{5000} \frac{1}{2}^{5000} \frac{1}{2}^{5000} = \binom{10000}{5000} \frac{1}{2}^{10000}$.

## 2 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn't perfect) before going to bed. This process then repeats.

(a) If Marcus has $n$ shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of $n$ involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of $n$ different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, *and* the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not throw any future shirts into that location and also does not consider it as a candidate for future dirtiest locations (it is too dirty). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of $n$ involving no summations.

**Solution:**

(a) The number of days that it takes for him to throw a shirt into the dirty pile can be represented as a geometric RV. For the first shirt, this is the geometric RV with $p = 1/n$. We can see this by noticing that every day the probability of getting the dirtiest shirt remains $1/n$.

We'll call $X_i$ the number of days that go until he throws the $i$th shirt into the dirty pile. Since on the $i$th shirt, there are $n-i+1$ shirts left, we get that $X_i \sim \text{Geometric}(1/(n-i+1))$. The number of days until he does his laundry is a sum of these variables. Therefore, we can get the following result:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} (n-i+1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b) For this part we can use a similar approach but the probability for $X_i$ becomes $1/(n-i+1)^2$. This is because the dirtiest shirt falls into the dirtiest spot with probability $1/(n-i+1)$ and we pick it after that with probability $1/(n-i+1)$, so the probability of picking the dirtiest shirt from the dirtiest spot for the $i$th shirt is $1/(n-i+1)^2$. Using the same approach, we get the following sum:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} (n-i+1)^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

## 3 Alternating Technicians

A faulty machine is repeatedly run and on each run, the machine fails with probability $p$ independent of the number of runs. Let the random variable $X$ denote the number of runs until the first failure. Now, two technicians are hired to check on the machine every run. They decide to take turns checking on the machine every run. What is the probability that the first technician is the first one to find the machine broken? (Your answer should a be closed-form expression.)

**Solution:**

Let the required probability be denoted by $q$ (the probability that the first technician is the first to find the machine fail). Now, we have:

$$q = \mathbb{P}[X = 1] + \mathbb{P}[X = 3] + \mathbb{P}[X = 5] + \dots,$$

because the first technician will be the first to find the machine broken if and only if the first failure occurs in an odd run. We are given that $X \sim \text{Geometric}(p)$ and hence

$$q = \sum_{k=0}^{\infty} \mathbb{P}[X = 2k+1] = \sum_{k=0}^{\infty} p(1-p)^{2k} = p\sum_{k=0}^{\infty}(1-2p+p^2)^k = \frac{p}{2p-p^2} = \frac{1}{2-p}.$$

Alternatively, suppose we decompose the above sum as follows:

$$\begin{aligned} q &= \mathbb{P}[X = 1] + \sum_{i=1}^{\infty} \mathbb{P}[\{X = 2i+1\} \cap \{X \neq 1\}] \\ &= \mathbb{P}[X = 1] + \sum_{i=1}^{\infty} \mathbb{P}[X \neq 1]\mathbb{P}[X = 2i+1 | X \neq 1] \end{aligned}$$

Using the memoryless property of the geometric distribution we have

$$\mathbb{P}[X = 2i+1 | X \neq 1] = \mathbb{P}[X = 2i+1 | X > 1] = \mathbb{P}[X = 2i],$$

and this further simplifies the above sum as follows:

$$\begin{aligned} q &= \mathbb{P}[X = 1] + \sum_{i=1}^{\infty} \mathbb{P}[X \neq 1]\mathbb{P}[X = 2i] \\ &= \mathbb{P}[X = 1] + \mathbb{P}[X \neq 1]\sum_{i=1}^{\infty} \mathbb{P}[X = 2i] \\ &= \mathbb{P}[X = 1] + \mathbb{P}[X \neq 1]\left(1 - \sum_{i=1}^{\infty} \mathbb{P}[X = 2i-1]\right) \\ &= \mathbb{P}[X = 1] + \mathbb{P}[X \neq 1](1-q) \end{aligned}$$

where the second last equality follows because the probability that $X$ is even is the complement of the event that $X$ is odd. The final equation is intuitive as in the event that the first technician doesn't find the machine broken in the first run, the memoryless property of the geometric distribution ensures that the probability that the second technician finds the machine broken first is the same as the probability that the first technician does when we have no knowledge of the first run. That is, we have:

$$q = \mathbb{P}[\text{Second technician finds the machines broken first} \mid \text{No machine fails in run 1}].$$

Using $\mathbb{P}[X = 1] = p$ and solving the above equation, we get that:

$$q = \frac{1}{2-p}.$$