1 Number Game

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0,100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V|S=s]$, where s is any constant such that $0 \le s \le 100$?

(c) What is $\mathbb{E}[V]$?

Solution:

(a) S is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.

(b) If $S = s$, we know that V will be uniformly distributed between s and 100. Similar to the previous part, this gives us that $\mathbb{E}[V|S=s] = \frac{s+100}{2}$.

(c) We have that

$$\begin{aligned}
\mathbb{E}[V] &= \sum_{s=0}^{100} \mathbb{E}[V|S=s] \cdot \mathbb{P}[S=s] \\
&= \sum_{s=0}^{100} \frac{s+100}{2} \cdot \frac{1}{101} \\
&= \frac{1}{202}\left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100\right)
\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

2 Joint Distributions

(a) Give an example of discrete random variables X and Y with the property that $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$. You should specify the joint distribution of X and Y.

(b) Give an example of discrete random variables X and Y that (i) are *not independent* and (ii) have the property that $\mathbb{E}[XY] = 0$, $\mathbb{E}[X] = 0$, and $\mathbb{E}[Y] = 0$. Again you should specify the joint distribution of X and Y.

Solution:

(a) Let $\mathbb{P}[X = 1] = \frac{1}{2}$, $\mathbb{P}[X = -1] = \frac{1}{2}$, and $Y \equiv X$. Then $\mathbb{E}[X] = 1\mathbb{P}[X = 1] + (-1)\mathbb{P}[X = -1] = 0$, and $\mathbb{E}[Y] = \mathbb{E}[X]$. Similarly, since $X = Y$, $\mathbb{E}[XY] = \mathbb{E}[X^2] = 1$ and $\mathbb{E}[X]\mathbb{E}[Y] = 0$.

(b) One example is given by $P(X = -1, Y = \frac{1}{3}) = P(X = 1, Y = \frac{1}{3}) = P(X = 0, Y = -\frac{2}{3}) = \frac{1}{3}$.

3 Inequality Practice

(a) X is a random variable such that $X \geq -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

(b) Y is a random variable such that $Y \leq 10$ and $\mathbb{E}[Y] = 1$. Find an upper bound for the probability of Y being less than or equal to -1.

(c) You roll a die 100 times. Let Z be the sum of the numbers that appear on the die throughout the 100 rolls. Compute $\text{Var}(Z)$. Then use Chebyshev's inequality to bound the probability of the sum Z being greater than 400 or less than 300.

Solution:

(a) We want to use Markov's Inequality, but recall that Markov's Inequality only works with non-negative random variables. So, we define a new random variable $\tilde{X} = X + 5$, where $\tilde{X}$ is always non-negative, so we can use Markov's on $\tilde{X}$. By linearity of expectation, $\mathbb{E}[\tilde{X}] = -3 + 5 = 2$. So, $\mathbb{P}[\tilde{X} \geq 4] \leq 2/4 = 1/2$.

(b) We again use Markov's Inequality. Similarly, define $\tilde{Y} = -Y + 10$, and $\mathbb{E}[\tilde{Y}] = -1 + 10 = 9$. $P[Y \leq -1] = P[-Y \geq 1] = P[-Y + 10 \geq 11] \leq 9/11$.

(c) Let Z_i be the number on the die for the ith roll, for $i = 1, \ldots, 100$. Then, $Z = \sum_{i=1}^{100} Z_i$. By linearity of expectation, $\mathbb{E}[Z] = \sum_{i=1}^{100} \mathbb{E}[Z_i]$.

$$\mathbb{E}[Z_i] = \sum_{j=1}^{6} j \cdot \mathbb{P}[Z_i = j] = \sum_{j=1}^{6} j \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j = \frac{1}{6} \cdot 21 = \frac{7}{2}$$

Then, we have $\mathbb{E}[Z] = 100 \cdot (7/2) = 350$.

$$\mathbb{E}[Z_i^2] = \sum_{j=1}^{6} j^2 \cdot \mathbb{P}[Z_i = j] = \sum_{j=1}^{6} j^2 \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j^2 = \frac{1}{6} \cdot 91 = \frac{91}{6}$$

Then, we have

$$\text{Var}(Z_i) = \mathbb{E}[Z_i^2] - \mathbb{E}[Z_i]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12},$$

Since the Z_is are independent, and therefore uncorrelated, we can add the $\text{Var}(Z_i)$s to get $\text{Var}(Z) = 100(35/12)$.

Finally, we note that we can upper bound $\mathbb{P}[|Z-350| > 50]$ with $\mathbb{P}[|Z-350| \geq 50]$.

Putting it all together, we use Chebyshev's to get

$$\mathbb{P}[|Z-350| > 50] < \mathbb{P}[|Z-350| \geq 50] \leq \frac{100(35/12)}{50^2} = \frac{7}{60}.$$