1 LLSE

We have two bags of balls. The fractions of red balls and blue balls in bag $A$ are $2/3$ and $1/3$ respectively. The fractions of red balls and blue balls in bag $B$ are $1/2$ and $1/2$ respectively. Someone gives you one of the bags (unmarked) uniformly at random. You then draw 6 balls from that same bag with replacement. Let $X_i$ be the indicator random variable that ball $i$ is red. Now, let us define $X = \sum_{1 \leq i \leq 3} X_i$ and $Y = \sum_{4 \leq i \leq 6} X_i$.

(a) Compute $E[X]$ and $E[Y]$.

(b) Compute $\text{Var}(X)$.

(c) Compute $\text{cov}(X, Y)$. (Hint: Recall that covariance is bilinear.)

(d) Now, we are going to try and predict $Y$ from a value of $X$. Compute $L(Y \mid X)$, the best linear estimator of $Y$ given $X$. (Hint: Recall that

$$L(Y \mid X) = E[Y] + \frac{\text{cov}(X, Y)}{\text{Var}(X)} (X - E[X]).$$

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2 Number Game

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0, 100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn’t like
losing, so he’s rigged his random number generator such that it instead picks randomly from the integers between Sinho’s number and 100. Let $S$ be Sinho’s number and $V$ be Vrettos’ number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V|S = s]$, where $s$ is any constant such that $0 \leq s \leq 100$?

(c) What is $\mathbb{E}[V]$?
3 Number of Ones

In this problem, we will revisit dice-rolling, except with conditional expectation.

(a) If we roll a die until we see a 6, how many ones should we expect to see?
(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

(Hint: for both of the above subparts, the Law of Total Expectation may be helpful)

4 Marbles in a Bag

We have $r$ red marbles, $b$ blue marbles, and $g$ green marbles in the same bag. If we sample marbles with replacement until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see? (Hint: It might be useful to use Law of Total Expectation, $E(Y) = E(E(Y|X))$.)