CS 70 Fall 2021 — Discrete Mathematics and Probability Theory — DIS 12B

1 Probabilistic Bounds

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}[X^2] = 13$.

(b) $\mathbb{P}[X = 2] > 0$.

(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$.

(d) $\mathbb{P}[X \leq 1] \leq 8/9$.

(e) $\mathbb{P}[X \geq 6] \leq 9/16$.

Solution:

(a) TRUE. Since $9 = \mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 2^2$, we have $\mathbb{E}[X^2] = 9 + 4 = 13$.

(b) FALSE. It is not necessary for a random variable to be able to take on its mean as a value. As one possible counterexample, construct a random variable X that satisfies the conditions in the question but does not take on the value 2. A simple example would be a random variable that takes on 2 values, where $\mathbb{P}[X = a] = \mathbb{P}[X = b] = 1/2$, and $a \neq b$ with both $a, b \leq 10$. The expectation must be 2, so we have $a/2 + b/2 = 2$. The variance is 9, so $\mathbb{E}[X^2] = 13$ (from Part (**??**)) and $a^2/2 + b^2/2 = 13$. Solving for a and b, we get $\mathbb{P}[X = -1] = \mathbb{P}[X = 5] = 1/2$ as a counterexample.

(c) FALSE. The median of a random variable is not necessarily the mean, unless it is symmetric. As one possible counterexample, construct a random variable X that satisfies the conditions in the question but does not have an equal chance of being less than or greater than 2. A simple example would be a random variable that takes on 2 values, where $\mathbb{P}[X = a] = p, \mathbb{P}[X = b] = 1 - p$. Here, we use the same approach as part (b) except with a generic p, since we want $p \neq 1/2$. The expectation must be 2, so we have $pa + (1 - p)b = 2$. The variance is 9, so $\mathbb{E}[X^2] = 13$ and $pa^2 + (1 - p)b^2 = 13$. Solving for a and b, we find the relation $b = 2 \pm 3/\sqrt{x}$, where $x = (1 - p)/p$. Then, we can find an example by plugging in values for x so that $a, b \leq 10$ and $p \neq 1/2$. One such counterexample is $\mathbb{P}[X = -7] = 1/10, \mathbb{P}[X = 3] = 9/10$.

(d) TRUE. Let $Y = 10 - X$. Since X is never exceeds 10, Y is a non-negative random variable. By Markov's inequality,

$$\mathbb{P}[10 - X \geq a] = \mathbb{P}[Y \geq a] \leq \frac{\mathbb{E}[Y]}{a} = \frac{\mathbb{E}[10 - X]}{a} = \frac{8}{a}.$$

Setting $a = 9$, we get $\mathbb{P}[X \leq 1] = \mathbb{P}[10 - X \geq 9] \leq 8/9$.

(e) TRUE. Chebyshev's inequality says $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \mathrm{Var}(X)/a^2$. If we set $a = 4$, we have

$$\mathbb{P}[|X - 2| \geq 4] \leq \frac{9}{16}.$$

Now we observe that $\mathbb{P}[X \geq 6] \leq \mathbb{P}[|X - 2| \geq 4]$, because the event $X \geq 6$ is a subset of the event $|X - 2| \geq 4$.

2 Working with the Law of Large Numbers

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

Solution:

(a) 10 tosses. By LLN, the sample mean should have higher probability to be close to the population mean as n increases. Therefore the average proportion of coins that are heads should be closer to 0.50, and has a lower chance of being greater than 0.60 if there are 100 tosses (compared with 10 tosses).

(b) 100 tosses. Again, by LLN, the sample mean should have higher probability to be close to the population mean as n increases. Therefore the average proportion of coins that are heads should be closer to 0.50, and has a lower chance of being smaller than 0.40 if there are 100 tosses. A lower chance of being smaller than 0.40 is the desired result.

(c) 100 tosses. Again, by LLN, the average proportion of coins that are heads should be closer to 0.50, and has a lower chance of being both smaller than 0.40 if there are 100 tosses. Similarly, there is a lower chance of being larger than 0.60 if there are 100 tosses. Lower chances of both of these events is desired if we want the fraction of heads to be between 0.4 and 0.6.

(d) 10 tosses. Compare the probability of getting equal number of heads and tails between $2n$ and $2n+2$ tosses.

$$\begin{aligned}
\mathbb{P}[n \text{ heads in } 2n \text{ tosses}] &= \binom{2n}{n}\frac{1}{2^{2n}} \\
\mathbb{P}[n+1 \text{ heads in } 2n+2 \text{ tosses}] &= \binom{2n+2}{n+1}\frac{1}{2^{2n+2}} = \frac{(2n+2)!}{(n+1)!(n+1)!}\cdot\frac{1}{2^{2n+2}} \\
&= \frac{(2n+2)(2n+1)2n!}{(n+1)(n+1)n!n!}\cdot\frac{1}{2^{2n+2}} \\
&= \frac{2n+2}{n+1}\cdot\frac{2n+1}{n+1}\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} < \left(\frac{2n+2}{n+1}\right)^2\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} \\
&= 4\binom{2n}{n}\cdot\frac{1}{2^{2n+2}} = \binom{2n}{n}\frac{1}{2^{2n}} = \mathbb{P}[n \text{ heads in } 2n \text{ tosses}]
\end{aligned}$$

As we increment n, the probability will always decrease. Therefore, the larger n is, the less probability we'll get exactly 50% heads. □

Note: By Stirling's approximation, $\binom{2n}{n}2^{-2n}$ is roughly $(\pi n)^{-1/2}$ for large n.

See https://github.com/dingyiming0427/CS70-demo/ for a code demo.

3 Continuous Computations

Let X be a continuous random variable whose PDF is cx^3 (for some constant c) in the range $0 \le x \le 1$, and is 0 outside this range.

(a) Find c.

(b) Find the CDF of X.

(c) Find $\mathbb{E}(X)$.

Solution:

(a) Since our total probability must be equal to 1,

$$\int_0^1 cx^3\,\mathrm{d}x = 1 = \frac{1}{4}cx^4\Big|_{x=0}^{1} = \frac{c}{4},$$

so $c = 4$.

(b) From Part (a), $c = 4$, so the PDF is $f(x) = 4x^3$ on the interval $[0,1]$ and is zero elsewhere. To find the CDF $F(x)$, we integrate $f(x0)$:

$$F(x) = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \begin{cases} 0 & x \le 0 \\ x^4 & 0 < x \le 1 \\ 1 & 1 < x \end{cases}$$

(c)

$$\mathbb{E}(X) = \int_0^1 x \cdot 4x^3 \,\mathrm{d}x = \int_0^1 4x^4 \,\mathrm{d}x = \left[\frac{4}{5}x^5\right]_{x=0}^1 = \frac{4}{5}.$$