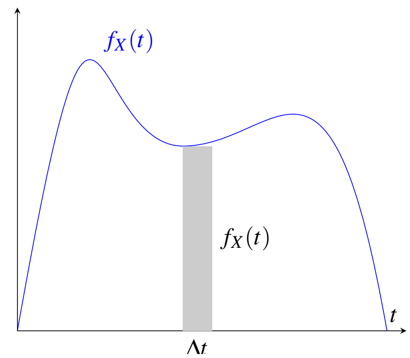
Continuous Probability Intro I

In discrete probability, we are only concerned with RVs that take on countably many values; now, in continuous probability, we are interested in RVs that take on *uncountably* many values. Here, the most important difference is that $\mathbb{P}[X = k] = 0$ for any k, but we can have $\mathbb{P}[X \in (a,b)] > 0$.

This gives the motivation for the **probability density function (PDF)**, denoted as $f_X(t)$.



Recall from physics that in 1D, density = mass/length; we define *probability density* similarly, as probability/length. In particular, this means that the area of the rectangle, $f_X(t)\Delta t$ (a product of density and length), is a probability. Generalizing this idea to find the area under the curve, we can now find probabilities from the PDF:

$$\mathbb{P}[a < X < b] = \int_a^b f_X(t)\,\mathrm{d}t.$$

From here, we define the **cumulative distribution function (CDF)** as

$$F_X(t) = \mathbb{P}[X < t] = \int_{-\infty}^{t} f_X(u)\,\mathrm{d}u.$$

From the fundamental theorem of calculus, we have $\frac{\mathrm{d}}{\mathrm{d}t}F_X(t) = f_X(t)$; the derivative of the CDF is the PDF.

Properties:

- $f_X(t) \geq 0$, and $\int_{-\infty}^{\infty} f_X(t)\,\mathrm{d}t = 1$
- $F_X(t)$ must be non-decreasing, with $\lim_{t\to\infty} F_X(t) = 1$ and $\lim_{t\to-\infty} F_X(t) = 0$

Other probability concepts follow naturally from these definitions; the only major difference from discrete probability is that sums turn into integrals, and $\mathbb{P}[X = t]$'s turn into $f_X(t)$'s. For example,

	Discrete	Continuous
Expectation	$\mathbb{E}[X] = \sum_t t \cdot \mathbb{P}[X = t]$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot f_X(t)\,\mathrm{d}t$
LOTUS	$\mathbb{E}[g(X)] = \sum_t g(t) \cdot \mathbb{P}[X = t]$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot f_X(t)\,\mathrm{d}t$
Total Probability	$\mathbb{P}[A] = \sum_{t=1}^{n} \mathbb{P}[A \mid X = t]\,\mathbb{P}[X = t]$	$\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A \mid X = t] f_X(t)\,\mathrm{d}t$

Exponential Distribution: $X \sim \text{Exponential}(\lambda)$, the continuous analog to the geometric distribution; it models the amount of time needed to wait until a success, where the rate of success is

λ.

$$f_X(t) = \lambda e^{-\lambda t} \qquad \mathbb{E}[X] = \frac{1}{\lambda}$$
$$F_X(t) = 1 - e^{-\lambda t} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Similar to the geometric distribution, the exponential distribution also has the memoryless property:

$$\mathbb{P}[X > m+n \mid X > m] = \mathbb{P}[X > n].$$

1 Continuous Intro

Note 21

(a) Is

$$g(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate the PDF $f_X(x)$, along with $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ if the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{\ell}, & 0 \le x \le \ell, \\ 1, & x \ge \ell \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

We will use these X, Y defined here for the rest of the subparts of this question. What is their joint density? (Hint: We can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the X and Y in part (c).

(e) Recall the definition of the joint distribution of X and Y: $\mathbb{P}[a \le X \le b, c \le Y \le d] = \int_a^b \int_c^d f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x$. Derive the marginal density function for continuous random variables (Hint: Start by computing the CDF $F_X(x)$):

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$$

(f) Let $Z = X + Y$ for the X and Y in part (c). Derive an expression for the joint distribution $f_{X,Z}(x,z)$ in terms of $f_X(x), f_Y(y)$ and use this to compute (but not evaluate) an integral expression for $f_Z(z)$.

Solution:

(a) Yes, it is a valid density function; it is non-negative and integrates to 1.

No, it is not a valid CDF; a CDF should go to 1 as x goes to infinity and be non-decreasing.

(b) We have

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} \frac{1}{\ell}, & 0 \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$
$$\mathbb{E}[X] = \int_{x=0}^{\ell} x \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell}{2}$$
$$\mathbb{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell^2}{3}$$
$$\mathrm{Var}(X) = \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{\ell^2}{12}$$

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted $\mathrm{Uniform}[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{P}[X \in [x, x+\mathrm{d}x], Y \in [y, y+\mathrm{d}y]] \\ &= \mathbb{P}[X \in [x, x+\mathrm{d}x]]\,\mathbb{P}[Y \in [y, y+\mathrm{d}y]] \\ &\approx f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint density is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) We have

$$\mathbb{E}[XY] = \int_{x=0}^{1}\int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = \frac{1}{3}.$$

Alternatively, since X and Y are independent, we can compute $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3},$$

and $\mathbb{E}[Y] = \frac{1}{2}$ since the density of Y is symmetric around $\frac{1}{2}$. Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3}.$$

(e)

$$\begin{aligned} \mathbb{P}[a \le X \le b] &= \mathbb{P}[a \le X \le b, -\infty \le Y \le \infty] \\ &= \int_a^b \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

Then, the CDF of X is $\mathbb{P}[\infty \le X \le x] = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x$. Using the FTC, we can differentiate both sides with respect to x: $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}[\infty \le X \le x] = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$.

(f) We note that in order for $Z = X + Y$, we have $X = X$, and $Y = Z - X$. Thus, we can write

$$\begin{aligned} f_{X,Z}(x,z) &= f_{X,Y}(x, z-x) \\ &= f_X(x) f_Y(z-x) \end{aligned}$$

Then, conditioning on x, we can write

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X,Z}(x,z) \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, \mathrm{d}x \\ &= \int_0^1 2x f_Y(z-x) \, \mathrm{d}x \\ &= \int_0^1 2x \cdot \mathbf{1}_{0 \le z-x \le 1} \, \mathrm{d}x \\ &= \int_0^1 2x \cdot \mathbf{1}_{z-1 \le x \le z} \, \mathrm{d}x \\ &= \int_{max(0,z-1)}^{min(1,z)} 2x \, \mathrm{d}x \end{aligned}$$

Alternatively, it would've been fine to condition on y instead of x, which would result in a different integral expression that would evaluate to the same number.

2 Darts Again

Note 21 Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim (but his throws may land outside of the dartboard); the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0,10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}[X < x]$.]

Solution: We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of X by solving for the CDF of X, $\mathbb{P}[X < x]$. To get this, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X < x] = \frac{\pi x^2}{\pi 10^2} = \frac{x^2}{100}.$$

for $x \in (0,10)$. For $x < 0$, the CDF is 0, and for $x > 10$, the CDF is 1.

Differentiating gives us the PDF of X, which is given by $f_X(x) = \frac{x}{50}$ for $x \in (0,10)$, and 0 elsewhere. Now, we solve for $\mathbb{P}[X < Y]$ with total probability:

$$\begin{aligned}\mathbb{P}[X < Y] &= \int_0^{10} \mathbb{P}[Y > X \mid X = x] f_X(x)\,\mathrm{d}x \\ &= \int_0^{10} \mathbb{P}[Y > x] f_X(x)\,\mathrm{d}x \\ &= \int_0^{10} e^{-0.5x} \frac{x}{50}\,\mathrm{d}x\end{aligned}$$

$\mathbb{P}[Y > x] = e^{-0.5x}$ comes from the (complement of the) exponential CDF. Evaluating this integral gives us $\mathbb{P}[X < Y] \approx 0.0767$.

Explanation of Integral: The integral may seem a bit confusing, but let's break it down. This is an expression of the total probability rule, where we condition on X. Recall that in discrete, we could calculate
$\mathbb{P}[X < Y] = \sum_x \mathbb{P}[X = x]\,\mathbb{P}[Y > x]$. In continuous, it is pretty much analogous, just with an integral and $f_X(x)\,\mathrm{d}x$ instead of the summation of $\mathbb{P}[X = x]$.

Alternative Calculation of PDF: Another way we could've calculate the PDF of X is by noticing that the PDF corresponds to the likelihood of falling on a point with radius x. The portion of the circle that corresponds to the radius of x is equivalent to its circumference, which is linear with respect to the radius, thus the likelihood (PDF) should be linear with respect to the radius as well. Thus, we have that $f_X(x) = cx$ for $x \in (0,10)$, and 0 elsewhere. The PDF must integrate to 1, so $\int_0^{10} cx\,\mathrm{d}x = 50c = 1$, which means that $c = \frac{1}{50}$.

Alternative Setup of Integral: You may have noticed that we chose to condition on X in our setup for total probability. It happens to be that this is the easier way to setup the integral, because the bounds are simpler (since X is bounded by 0 to 10) and $\mathbb{P}[Y > X \mid X = x]$ has a simple, continuous expression. We could've instead conditioned on Y, but it is more difficult. The interested reader may choose to follow along with the alternate integral setup as follows.

$$\begin{aligned}\mathbb{P}[X < Y] &= \int_0^{\infty} \mathbb{P}[X < Y \mid Y = y] f_Y(y)\,\mathrm{d}y \\ &= \int_0^{10} \frac{y^2}{100} \cdot 0.5e^{-y/2}\,\mathrm{d}y + \int_{10}^{\infty} 1 \cdot 0.5e^{-y/2}\,\mathrm{d}y\end{aligned}$$

This is because $\mathbb{P}[X < Y \mid Y = y]$ is the CDF of X: $F_X(y)$, which changes expression past $X = 10$. Thus, we must split the integral into two parts.

3 Lunch Meeting

Note 21

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving.

(a) Provide a sketch of the joint distribution of the arrival times of Alice and Bob. For which region of the graph will Alice and Bob actually meet?

(b) Based on your sketch, what is the probability that they will actually meet for lunch?

Solution:

(a) Let the random variable A be the time that Alice arrives and the random variable B be the time when Bob arrives. Since A and B are both uniformly distributed, it is helpful to visualize the distribution graphically. Consider Figure 1, plotting the space of all outcomes (a,b):

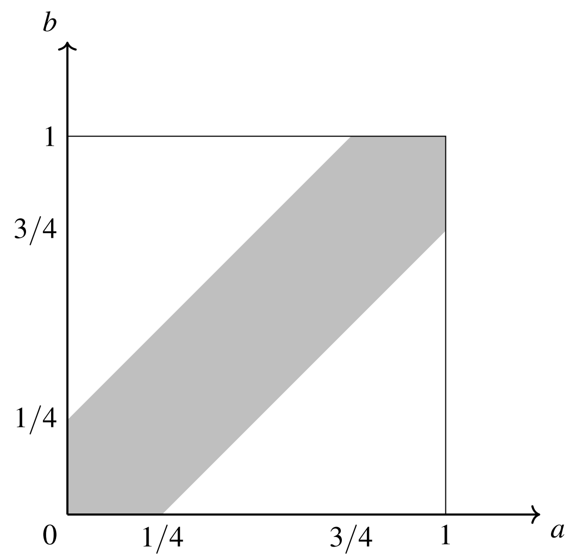


Figure 1: Visualization of joint probability density.

The arrival times are uniformly distributed over the box, and the shaded region is the set of values (a,b) for which Alice and Bob will actually meet for lunch.

(b) Since all points in this square are equally likely, the probably they meet is the ratio of the shaded area to the area of the square. If the area of the square is 1, then the area of the shaded region is

$$1 - 2 \times \left[\frac{1}{2} \times \left(\frac{3}{4}\right)^2\right] = \frac{7}{16},$$

since the area of the white triangle on the upper-left is $(1/2) \cdot (3/4)^2$, and the white triangle on the lower-right has the same area. Therefore, the probability that Alice and Bob actually meet is $7/16$.