

## Continuous Probability Intro II

**Normal (Gaussian) Distribution:**  $X \sim N(\mu, \sigma^2)$

The normal distribution occurs frequently in nature, mostly due to the Central Limit Theorem.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F_X(x) = \Phi(x)$$

Note that there is no closed form expression for the CDF of the normal distribution.

### Properties:

- A **standard normal** distribution is denoted as  $Z \sim N(0, 1)$
- If  $X \sim N(\mu, \sigma^2)$ , then  $\frac{X-\mu}{\sigma} \sim N(0, 1)$
- Generally, if  $X \sim N(\mu, \sigma^2)$ , then  $aX + b \sim N(a\mu + b, a^2\sigma^2)$
- If  $X \sim N(\mu_X, \sigma_X^2)$  and  $Y \sim N(\mu_Y, \sigma_Y^2)$  are independent, then

$$aX + bY \sim N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2).$$

**Central Limit Theorem:** Let  $X_1, X_2, \dots, X_n$  be i.i.d random variables with mean  $\mu$  and variance  $\sigma^2$ , and let

$$S_n = \sum_{i=1}^n X_i \quad A_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Note that

$$\mathbb{E}[A_n] = \mu \quad \text{Var}(A_n) = \frac{\sigma^2}{n}$$

The central limit theorem states that as  $n \rightarrow \infty$ ,  $A_n \rightarrow N(\mu, \frac{\sigma^2}{n})$ . Or,

$$S_n \rightarrow N(n\mu, n\sigma^2)$$

$$\frac{A_n - \mu}{\sigma/\sqrt{n}} \rightarrow N(0, 1)$$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \rightarrow N(0, 1)$$

These four equations are all equivalent formations of the same idea, which is that the sample mean (of i.i.d random variables) will converge to a normal distribution preserving the mean and variance of the sample mean, as  $n \rightarrow \infty$  (and the same holds for the sample sum, and shifted/scaled versions of them).

# 1 Interesting Gaussians

Note 21

- (a) If  $X \sim N(0, \sigma_X^2)$  and  $Y \sim N(0, \sigma_Y^2)$  are independent, then what is  $\mathbb{E}[(X+Y)^k]$  for any *odd*  $k \in \mathbb{N}$ ?
- (b) Let  $f_{\mu, \sigma}(x)$  be the density of a  $N(\mu, \sigma^2)$  random variable, and let  $X$  be distributed according to  $\alpha f_{\mu_1, \sigma_1}(x) + (1 - \alpha) f_{\mu_2, \sigma_2}(x)$  for some  $\alpha \in [0, 1]$ . Compute  $\mathbb{E}[X]$  and  $\text{Var}(X)$ . Is  $X$  normally distributed?

## Solution:

(a)  $\mathbb{E}[(X+Y)^k] = 0.$

Since  $X$  and  $Y$  are Gaussians, so must  $Z = X + Y$  be. Specifically,  $Z \sim N(0, \sigma_X^2 + \sigma_Y^2)$ . Thus, the PDF  $f_Z$  of  $Z$  is still symmetric about the origin; that is, it is an even function, i.e.  $f_Z(x) = f_Z(-x)$  for any  $a, b \in \mathbb{R}$ . Therefore,

$$\begin{aligned}\mathbb{E}[(X+Y)^k] &= \mathbb{E}[Z^k] = \int_{-\infty}^{\infty} x^k f_Z(x) dx \\ &= \int_{-\infty}^0 x^k f_Z(x) dx + \int_0^{\infty} x^k f_Z(x) dx \\ &= \int_0^{\infty} (-x)^k f_Z(-x) dx + \int_0^{\infty} x^k f_Z(x) dx \\ &= -\int_0^{\infty} x^k f_Z(x) dx + \int_0^{\infty} x^k f_Z(x) dx \\ &= 0,\end{aligned}$$

since  $k$  is odd.

Note that we could've just concluded that  $\int_{-\infty}^{\infty} x^k f_Z(x) dx = 0$  due to the fact that  $x^k f_Z(x)$  is an odd function (since  $x^k$  is an odd function for odd  $k$ ), and the integral from  $(-a, a)$  for any odd function will evaluate to 0.

Also note that adding two RVs is NOT equivalent to adding their PDFs. Instead, adding two RVs is equivalent to convolving their PDFs. As an example, for random variables  $X + Y = Z$ , it is true that  $f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx$ .

- (b)  $\mathbb{E}[X] = \alpha \mu_1 + (1 - \alpha) \mu_2$ ,  $\text{Var}(X) = \alpha(\sigma_1^2 + \mu_1^2) + (1 - \alpha)(\sigma_2^2 + \mu_2^2) - (\mathbb{E}[X])^2$ . No,  $X$  is not necessarily normally distributed.

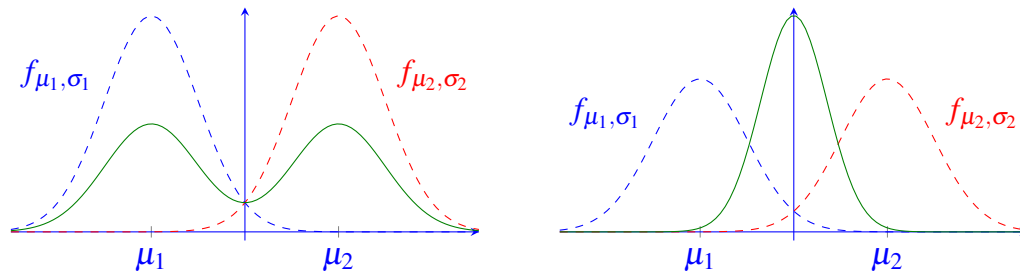
$$\begin{aligned}\mathbb{E}[X] &:= \mu = \int_{-\infty}^{\infty} x(\alpha f_{\mu_1, \sigma_1}(x) + (1 - \alpha) f_{\mu_2, \sigma_2}(x)) dx \\ &= \alpha \int_{-\infty}^{\infty} x f_{\mu_1, \sigma_1}(x) dx + (1 - \alpha) \int_{-\infty}^{\infty} x f_{\mu_2, \sigma_2}(x) dx = \alpha \mu_1 + (1 - \alpha) \mu_2 \\ \text{Var}(X) &:= \sigma^2 = \mathbb{E}[X^2] - \mu^2 = \alpha \int_{-\infty}^{\infty} x^2 f_{\mu_1, \sigma_1}(x) dx + (1 - \alpha) \int_{-\infty}^{\infty} x^2 f_{\mu_2, \sigma_2}(x) dx - \mu^2 \\ &= \alpha(\sigma_1^2 + \mu_1^2) + (1 - \alpha)(\sigma_2^2 + \mu_2^2) - \mu^2.\end{aligned}$$

We know that the density of  $N(\mu, \sigma)$  has a unique maximum at  $x = \mu$ ; however, if, e.g.  $\alpha = 1/2, \mu_1 = -10, \mu_2 = 10, \sigma_1 = \sigma_2 = 1$ , then  $\alpha f_{\mu_1, \sigma_1} + (1 - \alpha) f_{\mu_2, \sigma_2}$  has two maxima, and so cannot be the density of a Gaussian.

*Explanation of integrals:*  $\int_{-\infty}^{\infty} x f_{\mu_1, \sigma_1}(x) dx$  becomes  $\mathbb{E}[X_1]$  for  $X_1$  with PDF  $f_{\mu_1, \sigma_1}(x)$ , which is  $\mu_1$  by definition.

$\int_{-\infty}^{\infty} x^2 f_{\mu_1, \sigma_1}(x) dx$  becomes  $\mathbb{E}[X_1^2]$  for  $X_1$  with PDF  $f_{\mu_1, \sigma_1}(x)$ .  $\mathbb{E}[X_1^2] = \text{Var}(X_1) + \mathbb{E}[X_1]^2 = \sigma_1^2 + \mu_1^2$  by definition.

Below are some plots illustrating the difference between a linear combination of Gaussian *densities* and a linear combination of Gaussian *random variables*.



The left plot depicts  $\alpha f_{\mu_1, \sigma_1}(x) + (1 - \alpha) f_{\mu_2, \sigma_2}(x)$  in solid green. The right plot depicts the density of  $Z = \alpha Z_1 + (1 - \alpha) Z_2$  in solid green, where  $Z_1 \sim N(\mu_1, \sigma_1^2)$ , and  $Z_2 \sim N(\mu_2, \sigma_2^2)$ ; in particular, we have that  $Z \sim N(\alpha \mu_1 + (1 - \alpha) \mu_2, \alpha^2 \sigma_1^2 + (1 - \alpha)^2 \sigma_2^2)$ . (For simplicity, we fix  $\alpha = \frac{1}{2}$  in the plots.)

## 2 Binomial Concentration

Note 21

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to  $\infty$ . Suppose we have i.i.d. trials, each with a probability of success  $1/2$ . Let  $S_n$  be the number of successes in the first  $n$  trials ( $n$  is a positive integer).

- Compute the mean and variance of  $S_n$ .
- How should we define  $Z_n$  in terms of  $S_n$  to ensure that  $Z_n$  has mean 0 and variance 1?
- What is the distribution of  $Z_n$  as  $n \rightarrow \infty$ ?
- Use the bound  $\mathbb{P}[Z > z] \leq (\sqrt{2\pi}z)^{-1} e^{-z^2/2}$  when  $Z$  is a standard normal in order to approximately bound  $\mathbb{P}[S_n/n > 1/2 + \delta]$ , where  $\delta > 0$ .

**Solution:**

- Since  $S_n \sim \text{Binomial}(n, \frac{1}{2})$ , we have  $\mathbb{E}[S_n] = \frac{n}{2}$  and  $\text{Var}(S_n) = \frac{n}{4}$ .
- We can define

$$Z_n := \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}(S_n)}} = \frac{S_n - n/2}{\sqrt{n/2}}.$$

In particular, we subtract the mean and divide by the standard deviation to normalize  $S_n$ .

To check, we have

$$\begin{aligned}\mathbb{E}[Z_n] &= \frac{1}{\sqrt{n}/2} \mathbb{E}\left[S_n - \frac{n}{2}\right] = \frac{1}{\sqrt{n}/2} \left(\mathbb{E}[S_n] - \frac{n}{2}\right) = 0, \\ \text{Var}(Z_n) &= \frac{1}{n/4} \text{Var}\left(S_n - \frac{n}{2}\right) = \frac{1}{n/4} \text{Var}(S_n) = 1,\end{aligned}$$

since  $S_n \sim \text{Binomial}(n, 1/2)$ .

(c) The central limit theorem tells us that  $Z_n \rightarrow \mathcal{N}(0, 1)$ .

(d) In order to apply the bound, we must apply it to  $Z_n$ .

$$\begin{aligned}\mathbb{P}\left[\frac{S_n}{n} > \frac{1}{2} + \delta\right] &= \mathbb{P}\left[\frac{S_n - n/2}{n} > \delta\right] = \mathbb{P}\left[\frac{S_n - n/2}{\sqrt{n}/2} > 2\delta\sqrt{n}\right] \approx \mathbb{P}[Z_n > 2\delta\sqrt{n}] \\ &\leq \frac{1}{2^{3/2}\delta\sqrt{\pi n}} e^{-2\delta^2 n}\end{aligned}$$

### 3 Erasures, Bounds, and Probabilities

Note 21

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is  $p$ , and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most  $10^{-6}$ .

- (a) Use Chebyshev's inequality to upper bound  $p$  such that the probability of a communications breakdown is at most  $10^{-6}$ .
- (b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for  $p$  such that the probability of a communications breakdown is at most  $10^{-6}$ .

You may use that  $\Phi^{-1}(1 - 10^{-6}) \approx 4.753$ .

#### Solution:

- (a) Let  $X$  be the random variable denoting the number of erasures. Chebyshev's inequality states the following:

$$\mathbb{P}[|X - \mu_X| \geq k] \leq \frac{\sigma_X^2}{k^2}.$$

This gives us the bound

$$\begin{aligned}\mathbb{P}[X \geq 200] &= \mathbb{P}[X - \mu_X \geq 200 - \mu_X] \\ &\leq \mathbb{P}[|X - \mu_X| \geq 200 - \mu_X] \\ &\leq \frac{\sigma_X^2}{(200 - \mu_X)^2}\end{aligned}$$

Since  $X \sim \text{Binomial}(1000, p)$ , we have  $\mu_X = 1000p$  and  $\sigma_X^2 = 1000p(1-p)$ . Substituting these values in, we have

$$\mathbb{P}[X \geq 200] \leq \frac{1000p(1-p)}{(200 - 1000p)^2} = \frac{p(1-p)}{40(1-5p)^2}.$$

To meet our objective, we just have to ensure that

$$\mathbb{P}[X \geq 200] \leq \frac{p(1-p)}{40(1-5p)^2} \leq 10^{-6},$$

which yields an upper bound of about  $3.998 \times 10^{-5}$  for  $p$ .

- (b) Let  $Y$  be equal to the fraction of erasures, i.e.  $\frac{X}{1000}$ . Using properties of expectation and variance, we can see that

$$\begin{aligned}\mathbb{E}[Y] &= p \\ \text{Var}(Y) &= \text{Var}(X) \cdot \frac{1}{1000^2} = \frac{p(1-p)}{1000}\end{aligned}$$

Therefore, by Central Limit Theorem, we can say that  $Y$  is roughly a normal distribution with that mean and variance. Since we are interested in the event that  $Y \geq 0.2$ , let's figure out how many standard deviations above the mean 0.2 is:

$$\frac{0.2 - p}{\sqrt{\frac{p(1-p)}{1000}}} = \frac{(0.2 - p)\sqrt{1000}}{\sqrt{p(1-p)}}.$$

Therefore, the probability that we get a failure should be approximately (by CLT),

$$1 - \Phi\left(\frac{(0.2 - p)\sqrt{1000}}{\sqrt{p(1-p)}}\right)$$

where  $\Phi$  is the CDF of a standard normal variable. Setting this to be at most  $10^{-6}$  gives us

$$\Phi\left(\frac{(0.2 - p)\sqrt{1000}}{\sqrt{p(1-p)}}\right) \geq 1 - 10^{-6}$$

And, since  $\Phi^{-1}(1 - 10^{-6}) \approx 4.753$ , we solve the inequality

$$\frac{(0.2 - p)\sqrt{1000}}{\sqrt{p(1-p)}} \geq 4.753$$

This yields that we need  $p \leq 0.1468$ .

Note that this gives quite a different value from the previous parts. This is because the Central Limit Theorem gives a much tighter approximation for tail events than Markov's and Chebyshev's. However, we can only apply the Central Limit Theorem because  $n$  is large.

Therefore, we do not need  $p$  to be so low to achieve a communication breakdown probability of  $10^{-6}$ . The other bounds required us to need a probability of on the order of  $10^{-5}$ , but here we realize that we only need it to be less than 0.1468. (The true bound is .1459.) Quite drastic!