

## 1 Markov Chains Intro I

Note 22

A **Markov chain** models an experiment with states, transitioning between states with some probability. A Markov chain is uniquely defined with the following variables:

- $\mathcal{X}$  is the set of possible states in the Markov chain. For this course, we'll only be working with Markov chains with a finite state space.
- $X_n$  is a random variable denoting the state of the Markov chain at timestep  $n$ .
- $P$  is the transition matrix. The element row  $i$  and column  $j$  in the matrix is defined as

$$P(i, j) = \mathbb{P}[X_{n+1} = j \mid X_n = i].$$

In particular, this is the probability that we transition from state  $i$  to state  $j$ .

- $\pi_0$  is the initial distribution; it is a row vector, where  $\pi_0(i) = \mathbb{P}[X_0 = i]$ . (Similarly,  $\pi_n$  is the distribution of states at timestep  $n$ ; we have  $\pi_n(i) = \mathbb{P}[X_n = i]$ .)

Markov chains also have the **Markov property**:

$$\mathbb{P}[X_{n+1} = j \mid X_n = i, X_{n-1} = a_{n-1}, \dots, X_0 = a_0] = \mathbb{P}[X_{n+1} = j \mid X_n = i].$$

That is, the next state depends only on the current state, and not on any prior states (this is also known as the memoryless property of Markov chains).

The **stationary distribution** (or the **invariant distribution**) of a Markov chain is the row vector  $\pi$  such that  $\pi P = \pi$ . (That is, transitioning does not change the distribution of states.)

**Irreducibility**: A Markov chain is *irreducible* if one can reach any state from any other state in a finite number of steps.

**Periodicity**: In an irreducible Markov chain, we define the *period* of a state  $i$  as

$$d(i) = \gcd\{n > 0 \mid P^n(i, i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}.$$

If  $d(i) = 1$  for all  $i$ , then a Markov chain is *aperiodic*. Otherwise, we say that the Markov chain is *periodic*.

**Fundamental Theorem of Markov Chains**: If a Markov chain is irreducible and aperiodic, then for any initial distribution  $\pi_0$ , we have that  $\pi_n \rightarrow \pi$  as  $n \rightarrow \infty$ , and  $\pi$  is the unique invariant distribution for the Markov chain.

- (a) Consider the transition matrix  $P$  of a Markov chain.
- (i) Is it always true that every *row* of  $P$  sums to the same value? If so, state this value and briefly explain why this makes sense. If not, briefly explain why.
  
  - (ii) Is it always true that every *column* of  $P$  sums to the same value? If so, state this value and briefly explain why this makes sense. If not, briefly explain why.
- (b) Compute  $\mathbb{P}[X_1 = j]$  in terms of  $\pi_0$  and  $P$ . Then, express your answer in matrix notation—that is, give an expression for the row vector  $\pi_1$ , where  $\pi_1(j) = \mathbb{P}[X_1 = j]$ . Generalize your answer to express  $\pi_n$  in matrix form in terms of  $n$ ,  $\pi_0$ , and  $P$ .
- (c) Note that we only need to provide  $\mathcal{X}$ ,  $P$ , and  $\pi_0$  in order to uniquely define a Markov chain; the random variables  $X_n$  are implicitly defined.
- (i) Explain how you can compute the distributions of the random variables  $X_n$  for  $n \geq 0$  using only these parameters. (*Hint*: Part (b) can be helpful.)
  
  - (ii) The Markov property is also implicit in this definition of a Markov chain. If the Markov property *does not hold*, are  $\mathcal{X}$ ,  $P$ , and  $\pi_0$  sufficient to compute the distributions of  $X_n$  for  $n \geq 0$ ? Justify your answer.

## 2 Can it be a Markov Chain?

Note 22

- (a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and  $m$  and if the fly lands in either of those positions it is captured.

Given that the fly starts at state  $i$ , where  $1 < i < m$ , model this process as a Markov Chain. (Don't forget to specify the initial distribution!)

- (b) Take the same scenario as in the previous part with  $m = 4$ . Let  $Y_n = 0$  if at time  $n$  the fly is in position 1 or 2 and let  $Y_n = 1$  if at time  $n$  the fly is in position 3 or 4. Is the process  $Y_n$  a Markov chain?

### 3 Allen's Umbrella Setup

Note 22

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring exactly one umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is  $p$ .

(a) Model this as a Markov chain. What is  $\mathcal{X}$ ? Write down the transition matrix. (*Hint*: You should have 3 states. Keep in mind that our goal is to construct a Markov chain to solve part (c).)

(b) Determine if the distribution of  $X_n$  converges to the invariant distribution, and compute the invariant distribution.

(c) In the long term, what is the probability that Allen walks through rain with no umbrella?