Discussion 1B

CS 70, Summer 2024

1 Rationals and Irrationals

(a) Define $Q(x)$ as $(\exists p, q \in \mathbb{Z}) \left(q \neq 0 \land x = \frac{p}{q} \right)$.

Define $I(x)$ as $(x \in \mathbb{R}) \land (\forall p, q \in \mathbb{Z}) \left(q \neq 0 \implies x \neq \frac{p}{q} \right)$.

(b) $(\forall x, y \in \mathbb{R}) \Big(\big(Q(x) \land I(y) \land x \neq 0 \big) \implies I(xy) \Big)$.

(c) By contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Because they are rational, neither a nor b are zero; this means that c is also nonzero. So let $a = p/q$ and $c = r/s$, where p, q, r, s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

so b is rational. This is a contradiction, so we conclude that the product of a nonzero rational number and an irrational number is irrational.

2 Pebbles

(a) By direct proof. If there exists one column that has only red pebbles, no matter how you pick one pebble from each column, the one you pick from the red column is always going to be red.

(b) By contraposition. Suppose there does not exist a column with only red pebbles. This means that we can always find a blue pebble in each column. If we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. That is, there exists a way of choosing one pebble from each column without any red pebbles among the chosen pebbles.

3 Numbers of Friends

(a) From 0 to $n - 1$.

(b) No. Any partygoer can have at most $n - 2$ friends.

(c) By part **(a)**, each partygoer must have between 0 and $n - 1$ friends at the party. That's n total options for the number of friends each partygoer could have at the party. However, by part **(b)**, there cannot be simultaneously be a partygoer with 0 friends at the party and a partygoer with $n - 1$ friends at the party.

Therefore there are only $n - 1$ options for the number of friends that a partygoer could have at the party. By the pigeonhole principle, since there are n people at the party but only $n - 1$ options for the number of friends those people could have at the party, at least two partygoers must have the same number of friends.

4 Twin Primes

(a) First we note that any integer can be written in one of the forms $3k$, $3k + 1$, or $3k + 2$. (Note that $3k + 2$ is equal to $3(k + 1) - 1$. Since k is arbitary, we can treat these as equivalent forms).

Let $m > 3$ be arbitrary. We proceed by contraposition. Suppose that $m = 3k$ for some $k \in \mathbb{Z}$. Then m is divisible by 3, so it is composite.

(b) Checking by hand yields that 5 is the only prime up to 5 which participates in two twin prime pairs. For any prime $p > 5$, we must have that $p + 2$ and $p - 2$ are both prime for p to participate in two twin prime pairs. Since $p > 5$, we can apply part **(a)**. Either p is of the form $3k + 1$ or $3k - 1$.

(1) $p = 3k + 1$ for some $k \in \mathbb{Z}$. Then $p + 2 = 3k + 3$. Then 3 divides $p + 2$, so $p + 2$ is not prime.

(2) $p = 3k - 1$ for some $k \in \mathbb{Z}$. Then $p - 2 = 3k - 3$. Then 3 divides $p - 2$, so $p - 2$ is not prime.

In either case, at least one of $p + 2$ and $p - 2$ is not prime.