Discussion 2B

CS 70, Summer 2024

1 Countability Basics

(a) Yes. By contraposition. Let $m, n \in \mathbb{N}$ and suppose that $f(m) = f(n)$.

$$\begin{aligned} f(m) = f(n) &\implies m^2 = n^2 \\ &\implies m^2 - n^2 = 0 \\ &\implies (m+n)(m-n) = 0 \\ &\implies m = \pm n. \end{aligned}$$

If $n = 0$, then $+n = -n = m$. If $n \neq 0$, then $-n < 0$. Since $m \geq 0$, we must have that $m = +n$. Either way, $m = n$, as desired.

(b) Yes. For any $y \in \mathbb{R}$, we have that for $x = \sqrt[3]{y-1}$, $g(x) = y$.

(c) We construct injections $f : (0,1) \to (0,\infty)$ and $g : (0,\infty) \to (0,1)$.

Consider $f : (0,1) \to (0,\infty)$ given by

$$f(x) = x.$$

Then if $x \neq y$, $f(x) = x \neq y = f(y)$. So f is an injection.

Consider $g : (0,\infty) \to (0,1)$ given by

$$g(x) = \frac{1}{x+1}.$$

We show that g is an injection by contraposition. Suppose that $g(x) = g(y)$. Then

$$\frac{1}{x+1} = \frac{1}{y+1} \implies x = y.$$

Hence g is injective.

Thus we have an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$. By the Bernstein-Schroder theorem there exists a bijection from $(0,1)$ to $(0,\infty)$, and hence they have the same cardinality.

2 Unions and Intersections

(a) Always countable. $A \cap B$ is a subset of A, which is countable.

(b) Always uncountable. $A \cup B$ is a superset of B, which is uncountable.

(c) Sometimes countable, sometimes uncountable.

When the S_i are disjoint, the intersection is empty, and thus countable. For example, let

$$A = \mathbb{N} \qquad \text{and let} \qquad S_i = \{i\} \times \mathbb{R} = \{(i,x) \mid x \in \mathbb{R}\}.$$

Then $\bigcap_{i \in A} S_i = \varnothing$, which is countable.

However, when the S_i are identical, the intersection is uncountable. Let $A = \mathbb{N}$, let $S_i = \mathbb{R}$ for all i.

Then $\bigcap_{i \in A} S_i = \mathbb{R}$ is uncountable.

3 Countability Proof Practice

(a) Countable. Each disk must contain at least one rational point (an (x,y)-coordinate where $x, y \in \mathbb{Q}$) in its interior.

Due to the fact that no two disks overlap, the cardinality of the set of disks can be no larger than the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable.

(b) Possibly uncountable. Consider the circles $C_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$. For $r_1 \neq r_2$, C_{r_1} and C_{r_2} do not overlap, and there are uncountably many of these circles (one for each real number).

4 Counting Functions

(a) Uncountable. Assume for contradiction that such functions are countable. Then can enumerate them as

	0	1	2	3	$\dots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$\dots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$\dots$
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$\dots$
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Define f such that $f(n) > f(n-1)$ and $f(n) > f_n(n)$ for each $n \in \mathbb{N}$. This function is non-decreasing by construction, but differs from each f_i and therefore cannot be on the list. This contradicts the fact that the list enumerated all non-decreasing functions. Therefore there must be uncountably many such functions.

(b) Countable. Let F be the set of non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$ and let $D_n \subseteq F$ be the subset of non-increasing functions which start at n, i.e., $f(0) = n$. If such a function decreased more than n times, it would fall below 0. Therefore every such function can only decrease at most n points.

For $f \in D_n$, suppose it decreases $k \leq n$ times. Let $x_1, \dots, x_k$ be the points at which f decreases and let y_i be the amount by which f decreases at x_i. Then f is uniquely described by

$$((x_1, y_1), \dots, (x_k, y_k) \in (\mathbb{N}^2)^k.$$

We can pad this description with $n - k$ zeros to get

$$((x_1, y_1), \dots, (x_k, y_k), (0,0), \dots, (0,0)) \in (\mathbb{N}^2)^n.$$

We have shown an injection from D_n to $(\mathbb{N}^2)^n$, so $|D_n| \leq |(\mathbb{N}^2)^n|$. We have seen that $\mathbb{N}^2$ is countable. We can show by induction that $(\mathbb{N}^2)^n$ is countable. Therefore D_n is countable.

Note that

$$F = \bigcup_{n \in \mathbb{N}} D_n.$$

We can construct an injection from F to $\mathbb{N} \times \mathbb{N}$ as follows. For each $n \in \mathbb{N}$, let $\{d_n^{(1)}, d_n^{(2)}, \dots\}$ be the enumeration of D_n. The following table displays an injection of F into $\mathbb{N} \times \mathbb{N}$.

	0	1	2	3	$\dots$
D_0	$d_0^{(0)}$	$d_0^{(1)}$	$d_0^{(2)}$	$d_0^{(3)}$	$\dots$
D_1	$d_1^{(0)}$	$d_1^{(1)}$	$d_1^{(2)}$	$d_1^{(3)}$	$\dots$
D_2	$d_2^{(0)}$	$d_2^{(1)}$	$d_2^{(2)}$	$d_2^{(3)}$	$\dots$
D_3	$d_3^{(0)}$	$d_3^{(1)}$	$d_3^{(2)}$	$d_3^{(3)}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Since $\mathbb{N} \times \mathbb{N}$ is countable, so is F.

(c) Uncountable. We can inject the set of infinitely long binary strings into the set of bijective functions from $\mathbb{N}$ to $\mathbb{N}$. For any binary string $b = \{b_0, b_1, b_2, \dots\}$, consider the function $f : \mathbb{N} \to \mathbb{N}$ given by

$$\begin{aligned} f(2n) = 2n \text{ and } f(2n+1) = 2n+1 \qquad & \text{if } b_n = 0, \\ f(2n) = 2n+1 \text{ and } f(2n+1) = 2n \qquad & \text{if } b_n = 1. \end{aligned}$$

Note that f is a bijection. Since f is uniquely defined by the binary string b, the mapping from infintely long binary strings to bijective functions is injective.

Since the set of infinitely long binary strings is uncountable, and we produced an injection from that set to the set of bijective functions on $\mathbb{N}$, that set must be uncountable as well.