Discussion 2B

CS 70, Summer 2024

1 Countability Basics

(a) Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = n^2$. Is f an injection? Briefly justify your answer.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x^3 + 1$. Is g a surjection? Briefly justify your answer.

(c) The Bernstein-Schroder theorem states that, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then a bijection exists between A and B.

Use this to demonstrate that the unit interval $(0, 1)$ and the positive reals $\mathbb{R}_+ = (0, \infty)$ have the same cardinality.

2 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples—one where the expression is countable, and one where the expression is uncountable.

(a) $A \cap B$, where A is countable and B is uncountable.

(b) $A \cup B$, where A is countable and B is uncountable.

(c) $\bigcap_{i \in A} S_i$, where A is a countable set of indices and each S_i is an uncountable set.

3 Countability Proofs

(a) A disk is a 2D region of the form $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap (with possibly varying x_0, y_0, and r values).

Is this set always countable, or potentially uncountable?

(*Hint*: Attempt to relate this set to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$.)

(b) A circle is a subset of the plane of the form $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$.

Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap (with possibly varying x_0, y_0, and r values).

Is this set always countable, or potentially uncountable?

(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

4 Counting Functions

Prove whether the following sets are countable or uncountable.

(a) The set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions $f : \mathbb{N} \to \mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

(c) The set of all bijections from $\mathbb{N}$ to $\mathbb{N}$.