Discussion 2D

CS 70, Summer 2024

1 Cube Dual

We define the *dual* of a planar graph $G = (V, E)$ to be the graph G' constructed by replacing each face in G with a vertex, with edges between pairs of vertices G' if their corresponding faces are adjacent in G.

Consider the three-dimensional hypercube H_3.

(a) Draw a planar representation of H_3 and the corresponding dual graph H_3'. Determine whether H_3' is planar.

(*Hint*: Consider how the dual graph is drawn using the planar representation of H_3.)

(b) Is H_3' bipartite?

2 Planarity and Coloring

Prove or disprove each of the following statements.

(a) There exists a graph with 9 edges and 5 vertices that is 4-colorable.

(b) There exists a graph with 9 edges and 5 vertices that is *not* 4-colorable.

(c) There exists a graph with 10 edges and 6 vertices that is 4-colorable.

(d) There exists a graph with 10 edges and 6 vertices that is *not* 4-colorable.

3 Touring Hypercube

In lecture, you have seen that if G is a hypercube of dimension n, then

- the vertices of G are the binary strings of length n, and
- two vertices u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- each vertex appears exactly once in the sequence,
- each pair of consecutive vertices is connected by an edge, and
- v_0 and v_k are connected by an edge.

(a) Prove that a hypercube of dimension n has an Eulerian tour if and only if n is even.

(b) Prove that every hypercube has a Hamiltonian tour.

4 Binary Trees

You may have seen the recursive definition of binary trees from previous classes. In this class, we define binary trees in graph theoretic terms as follows.

- A binary tree of height $h > 0$ is a tree where exactly one vertex, called the *root*, has degree 2, and all other vertices have degrees 1 or 3. Vertices with degree 1 are known as *leaves*. The *height* h is defined as the maximum length of any path between the root and a leaf.
- A binary tree of height $h = 0$ is the graph with a single vertex. This vertex is both a leaf and a root.

(a) Let T be a binary tree and let $h(T) > 0$ be its height. Let r be the root of T and u and v be its neighbors.

(i) Prove that removing r from T will result in two binary trees L and R, with roots u and v, respectively.

(ii) Prove that $h(t) = \max(h(L), h(R)) + 1$.

(b) Prove that the number of vertices in a binary tree of height h is at most $2^{h+1} - 1$.

(c) Prove that a binary trees with n leaves has $2n - 1$ vertices.