Discussion 4B

CS 70, Summer 2024

1 Counting Practice

(a) Each element has two choices: either to be in the set or not in the set.

By first rule of counting, there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ subsets.

Alternatively, we can count the number of subsets of size 0 to n. Then, there are

$$\binom{n}{0} + \ldots + \binom{n}{n} = 2^n \text{ subsets.}$$

(b) Any k-clique is uniquely specified by the k vertices. The number of ways to pick k vertices among n is $\binom{n}{k}$.

(c) The total number of any size clique is $2^n - 1$, since each subset of $\{v_1, \ldots, v_n\}$ corresponds to a distinct clique. However, we do not want to include 0-cliques, 1-cliques, or 2-cliques. Therefore, we subtract them, respectively, to get

$$2^n - 1 - \binom{n}{1} - \binom{n}{2}.$$

We do not have to worry about inclusion-exclusion, since none of the groups we are looking at overlap.

(d) Any integer solution must consist of three numbers which sum to n. This can be represented as a balls and bins problem! We can throw n balls (stars) into 3 bins (2 bars), where the number of balls in each bin correspond to the values of x, y, and z.

Now, to count the number of solutions, we can just count the number of ways to arrange the stars and bars:

$$\binom{n+2}{2} = \binom{n+2}{n}.$$

(e) To uniquely define a sequence of non-increasing digits, we just need to know how many times each digit occurs. For example, given a set of digits $\{0, 0, 1, 2, 2, 3\}$, there is only one way to create a sequence of non-increasing digits: 322100.

Thus, this becomes a balls and bins problem where the bins represent the digits 0 to 9 and the balls represent how many times the digit occurs. There are n balls and 10 bins, which also means there are n stars and 9 bars, so the total number of non-increasing n-digit sequences is:

$$\binom{n+9}{9} = \binom{n+9}{n}.$$

(f) There is only one way to create a strictly decreasing n-digit sequence given n digits. That means, we just need to count the number of ways to choose n digits from the ten available:

$$\binom{10}{n}.$$

(g) If there are no birthdays in winter, then each worker has only three options for their birthday's season: spring, summer, fall. By the second rule of counting, there are

$$3^n \text{ ways.}$$

(h) Let S_1, S_2, S_3, and S_4 be the configurations with no birthdays in winter, spring, summer, and fall, respectively. The configurations where there's at least one season with no birthdays is

$$S = S_1 \cup S_2 \cup S_3 \cup S_4.$$

By the Principle of Inclusion-Exclusion,

$$\begin{aligned} |S| = {} & |S_1| + |S_2| + |S_3| + |S_4| \\ & - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_1 \cap S_4| - |S_2 \cap S_3| - |S_2 \cap S_4| - |S_3 \cap S_4| \\ & + |S_1 \cap S_2 \cap S_3| + |S_1 \cap S_2 \cap S_4| + |S_1 \cap S_3 \cap S_4| + |S_2 \cap S_3 \cap S_4| \\ & - |S_1 \cap S_2 \cap S_3 \cap S_4|. \end{aligned}$$

We know the sets at each "level" are the same size by symmetry (e.g., we can't have that $|S_1 \cap S_2| \neq |S_3 \cap S_1|$). Then, we can simplify $|S|$ to

$$|S| = \binom{4}{1}|S_1| - \binom{4}{2}|S_1 \cap S_2| + \binom{4}{3}|S_1 \cap S_2 \cap S_3| - \binom{4}{4}|S_1 \cap S_2 \cap S_3 \cap S_4|.$$

We saw in part (**g**) that $|S_1| = 3^n$. Similarly, $|S_1 \cap S_2| = 2^n$, because if there are no birthdays in winter or spring, they must all by in summer or fall. Continuing this, $|S_1 \cap S_2 \cap S_3| = 1^n$ and $|S_1 \cap S_2 \cap S_3 \cap S_4| = 0^n$ (there are no ways where there are no birthdays in any of the seasons).

So,

$$|S| = \binom{4}{1}3^n + \binom{4}{2}2^n + \binom{4}{3}1^n + \binom{4}{4}0^n.$$

2 Casting Counting

(a) **LHS:** This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n. Then, we consider three cases:

(i) Choose 2 directors from group 1

(ii) Choose 2 directors from group 2

(iii) Choose 1 director from group 1 and 1 director from group 2

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, the sum counts the total number of ways to choose 2 directors out of the $2n$ candidates.

(b) **LHS:** This is the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members by splitting it up into two cases: accept the first candidate or not accept the first candidate.

(i) If Alyssa selects the first candidate, then Alyssa needs to choose $k-1$ more crew members from the remaining $n-1$ candidates.

(ii) If Alyssa does not select the first candidate, then Alyssa needs to choose k crew members from the remaining $n-1$ candidates.

The number of ways we can do each of these things is $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$, respectively. Since these cases are mutually exclusive and cover all possibilities, the sum counts the total number of ways to choose k crew members out of n candidates.

(c) In this part, Alyssa selects a subset of the n actors to be in her musical. Additionally, she must select one individual as the lead for her musical.

LHS: Alyssa casts k actors in her musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Alyssa selects one lead for her musical (note that $n = \binom{n}{1}$). Then, for the remaining $n-1$ actors, she decides whether or not she would like to include them in the cast for a total of 2^{n-1} subsets.

(d) In this part, Alyssa selects a subset of the n actors to be in the musical. Additionally, she selects j lead actors (instead of only 1 in the previous part).

LHS: Alyssa casts $k \geq j$ actors in her musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Alyssa would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Alyssa selects j leads for her musical. Then, for the remaining $n-j$ actors, she decides whether or not she would like to include them in the cast for a total of 2^{n-j} subsets.

3 A Totient Identity

(a) **Scenario:** The number of fractions in the set $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$.

LHS: We count the fractions in the set in their simplified form.

For each divisor d of n, consider the fraction $\frac{m}{n}$ in the set which simplifies to $\frac{k}{d}$ for some $k \in \mathbb{Z}$. Since this fraction is in reduced form, k and d must share no common factors and therefore $\gcd(k, d) = 1$. So $k \in S_d$.

Thus, the number of fractions which have denominator d is $\varphi(d)$. When we sum over all divisors of n, we sum over how many of each denominator appear in the list.

RHS: There are n fractions in the set.