Discussion 4D

CS 70, Summer 2024

1 Urns and Marbles

(a) If Lance sees both marbles, it's intuitively more likely that he picked Urn 2. This is because it's more likely to see both marbles when drawing from Urn 2 than it is when drawing from Urn 1.

(b) Let A be the event that Lance sees both colors of marbles and U_1 be the event that Lance chose Urn 1.

This is a situation where the probability $\mathrm{P}(A \mid U_1)$ is more intuitive to think about, but we want to reverse the direction of conditioning. That's our cue to use Bayes rule. Then

$$\mathrm{P}(U_1 \mid A) = \frac{\mathrm{P}(A \mid U_1)\mathrm{P}(U_1)}{\mathrm{P}(A)}.$$

To find the numerator, we can use the fact that A happens if and only if Lance either draws black then white or white then black. So

$$\mathrm{P}(A \mid U_1) = \mathrm{P}(\mathrm{WB} \mid U_1) + \mathrm{P}(\mathrm{BW} \mid U_1) = \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} = 2 \cdot \frac{3}{4} \cdot \frac{1}{4}.$$

To find the denominator, we can partition on whether Lance chose Urn 1 or Urn 2.

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(A \cap U_1) + \mathrm{P}(A \cap U_2) \\ &= \mathrm{P}(A \mid U_1)\mathrm{P}(U_1) + \mathrm{P}(A \mid U_2)\mathrm{P}(U_2) \\ &= \left(2 \cdot \frac{3}{4} \cdot \frac{1}{4}\right)\frac{1}{2} + \left(2 \cdot \frac{2}{4} \cdot \frac{2}{4}\right)\frac{1}{2}.\end{aligned}$$

Therefore

$$\mathrm{P}(U_1 \mid A) = \frac{\left(2 \cdot \frac{3}{4} \cdot \frac{1}{4}\right)\frac{1}{2}}{\left(2 \cdot \frac{3}{4} \cdot \frac{1}{4}\right)\frac{1}{2} + \left(2 \cdot \frac{2}{4} \cdot \frac{2}{4}\right)\frac{1}{2}} = \frac{3}{3+4} = \frac{3}{7} < \frac{1}{2}.$$

We can see that Urn 2 is indeed more likely to have been chosen if Lance saw both colors.

(c) We can do this by partitioning on which urn Lance chose.

$$\mathrm{P}(B_1) = \mathrm{P}(B_1 \mid U_1)\mathrm{P}(U_1) + \mathrm{P}(B_1 \mid U_2)\mathrm{P}(U_2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{4} \cdot \frac{1}{2} = \frac{3}{8}.$$

(d) By symmetry, the chance that Lance draws a white or a black marble on the second draw should be the same as the first draw. In particular,

$$\mathrm{P}(W_2) = 1 - \mathrm{P}(B_1) = 1 - \frac{1}{4} \cdot \frac{1}{2} + \frac{2}{4} \cdot \frac{1}{2} = \frac{5}{8}.$$

Of course, we can reach the same answer by partitioning on which urn was chosen:

$$\mathrm{P}(W_2) = \mathrm{P}(W_2 \cap U_1) + \mathrm{P}(W_2 \cap U_2).$$

(e) By partitioning on which urn was chosen, we have that

$$\mathrm{P}(B_1 W_2) = \mathrm{P}(B_1 W_2 \mid U_1)\mathrm{P}(U_1) + \mathrm{P}(B_1 W_2 \mid U_2)\mathrm{P}(U_2) = \left(\frac{1}{4} \cdot \frac{3}{4}\right)\frac{1}{2} + \left(\frac{2}{4} \cdot \frac{2}{4}\right)\frac{1}{2} = \frac{1}{2} \cdot \frac{7}{16} = \frac{7}{32}.$$

Note that this is not the same as

$$\mathrm{P}(B_1)\mathrm{P}(W_2) = \frac{3}{8} \cdot \frac{5}{8} = \frac{15}{64}.$$

So the two events are not independent.

2 The Matching Problem

In a house of n cats, each cat has a designated bowl with their name on it. However, when it comes time for dinner, each cat select a bowl uniformly at random, independently of the choices of all the other cats.

(a) The cat has n equally likely options, of which one is correct. So the chance is $1/n$.

(b) Let M_i be the event that the i^{th} cat selects the right bowl.

$$P(M_1 \cap M_2) = P(M_1)P(M_2 \mid M_1) = \frac{1}{n} \cdot \frac{1}{n-1},$$

where $P(M_2) = 1/(n-1)$ since once we have that the first cat chooses the right bowl, the second cat has $n-1$ equally likely options of which one is correct.

(c) If we try working out the intersection, we see that the dependence structure is too complicated. So we take the complement and instead find the chance that at least one cat selects the right bowl.

That's the chance of $M_1 \cup M_2 \cup \ldots \cup M_n$. We can't use the addition rule because the events aren't mutually exclusive. So we need to use inclusion exclusion.

$$P\left(\bigcup_{i=1}^{n} M_i\right) = \sum_i P(M_i) - \sum_{i,j} P(M_i \cap M_j) + \ldots$$

We have already seen that $P(M_i) = 1/n$ and $P(M_i \cap M_j) = 1/n(n-1)$. Continuing this, $P(M_i \cap M_j \cap M_k) = 1/n(n-1)(n-2)$, and so on.

Therefore each summation is summing a constant term. So we can replace each sum by that term times the number of terms in the sum. The k^{th} sum has $\binom{n}{k}$ terms, so we get that

$$P\left(\bigcup_{i=1}^{n} M_i\right) = \binom{n}{1}\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} - \ldots + (-1)^{n-1}\binom{n}{n}\frac{1}{n!}.$$

That is,

$$P\left(\bigcup_{i=1}^{n} M_i\right) = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k}\frac{(n-k)!}{n!}.$$

3 Bag of Marbles

Timothe has a bag of n marbles, of which r are red, b are blue, and g are green. Timothe draws marbles from the bag without replacement.

(a) There are n equally likely outcomes, of which r are red. The chance is r/n.

(b) Again, there are n equally likely outcomes, of which r are red. The chance is r/n.

This isn't convincing to students. We can do it by partitioning as well:

$$P(R_2) = P(R_2R_1) + P(R_2R_1^{C}) = \frac{r-1}{n-1} \cdot \frac{r}{n} + \frac{r}{n-1} \cdot \frac{n-r}{n} = \frac{r}{n}\left(\frac{r-1}{n-1} + \frac{n-r}{n-1}\right) = \frac{r}{n}.$$

(c) Again, there are n equally likely outcomes, of which r are red. The chance is r/n.

The students won't be convinced, but we can't work this one out by partitioning on every possible sequence up to the 10th marble. Instead, we'll do the following.

Consider the entire sequence of all n draws, until all the marbles have been removed. There are $r \cdot (n-1)!$ of them which have a red marble at the 10^{th} position. There are $n!$ total sequences. So the chance that such a sequence has a red marble at the tenth position is

$$\frac{r(n-1)!}{n!} = \frac{r}{n},$$

as we knew.

(d) If we know that the fourth marble is blue, then there are $n-1$ equally likely outcomes, of which r of them are red. So the chance is

$$\frac{r}{n-1}.$$

Again, students may not be convinced. We can use a counting argument instead. There are $b(n-1)!$ sequences where the fourth marble is blue. Among those, $rb(n-2)!$ also have a red marble in the first position. So the chance is

$$\frac{rb(n-2)!}{b(n-1)!} = \frac{r}{n-1}.$$

(e) Each set of ten marbles is equally likely. There are $\binom{n}{10}$ ways to choose ten marbles from the n total. $\binom{r}{4}\binom{n-r}{6}$ of them have exactly 4 red marbles. So the chance is

$$\frac{\binom{r}{4}\binom{n-r}{6}}{\binom{n}{10}}.$$