Discussion 5B

CS 70, Summer 2024

1 Conditional Probability

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\Pr(R \mid W) = 0.3$, $\Pr(R \mid \overline{W}) = 0.8$ and $\Pr(W) = 0.2$. Then probability that a given day is both rainy and windy is $\Pr(R \cap W) = \Pr(R \mid W)\Pr(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\Pr(R) = \Pr(R \mid W)\Pr(W) + \Pr(R \mid \overline{W})\Pr(\overline{W}) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the weather on the first day doesn't affect that of the second, $\Pr(R_1) = \Pr(R_2) = \Pr(R)$. The required probability is then just $\Pr(R_1 \cap \overline{R_2}) + \Pr(\overline{R_1} \cap R_2) = \Pr(R_1)\Pr(\overline{R_2}) + \Pr(\overline{R_1})\Pr(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$. Since the weather on the first day does not affect the weather on the second day we can multiply the probabilities.

2 Random Variable

(a) The marginal distribution of X is $\Pr[X = 0] = 2/3$, $\Pr[X = 1] = 1/9, \Pr[X = 2] = 2/9$.

The marginal distribution of Y is $\Pr[Y = 0] = 4/9, \Pr[Y = 1] = 2/9, \Pr[Y = 2] = 1/3$.

(b) The conditional distribution of X conditioning on $Y = 0$ is $\Pr[X = 0|Y = 0] = 3/4, \Pr[X = 1|Y = 0] = 0, \Pr[X = 2|Y = 0] = 1/4$.

(c) The conditional distribution of X conditioning on $1 \leq X + Y \leq 2$ is $\Pr[X = 0|1 \leq X + Y \leq 2] = 3/5, \Pr[X = 1|1 \leq X + Y \leq 2] = 1/5, \Pr[X = 2|1 \leq X + Y \leq 2] = 1/5$.

3 Mutually Independent Events

(a) 2/5.

(b) $\Pr[A \cap B] = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$ and $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] = \frac{2}{5} + \frac{3}{5} - \frac{6}{25} = \frac{19}{25}$.

(c) $\Pr[A \cap B \cap C] = \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{10} = \frac{9}{125}$. There are 2 ways to calculate $\Pr[A \cup B \cup C]$:

First: Using inclusion-exclusion, we have

$$\begin{aligned}\Pr[A \cup B \cup C] &= \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[A \cap C] + \Pr[A \cap B \cap C] \\ &= \frac{2}{5} + \frac{3}{5} + \frac{3}{10} - \frac{2}{5} \cdot \frac{3}{5} - \frac{3}{5} \cdot \frac{3}{10} - \frac{2}{5} \cdot \frac{3}{10} + \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{10} = \frac{104}{125}\end{aligned}$$

Second: Using complement event, we have

$$\Pr[A \cup B \cup C] = 1 - \Pr[\neg A \cap \neg B \cap \neg C] = 1 - \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{7}{10} = \frac{104}{125}.$$

4 Working with Distributions

(a) The sample space of Y includes the values: $0, 1, 2, 3, 4, 5$.

$$\Pr(Y = 0) = \frac{1}{32}$$

$$\Pr(Y = 1) = \frac{5}{32}$$

$$\Pr(Y = 2) = \frac{10}{32}$$

$$\Pr(Y = 3) = \frac{10}{32}$$

$$\Pr(Y = 4) = \frac{5}{32}$$

$$\Pr(Y = 5) = \frac{1}{32}$$

(b) In every trial, the probability that the value observed is less than 3 is $1/3$. Therefore, we can think of each trial as a Bernoulli experiment where the success probability, $p = 1/3$, and we therefore get that N is a Geometric random variable.

Thus the probabilities can be expressed as:

$$\Pr(N = k) = \frac{1}{3}\left(\frac{2}{3}\right)^{k-1}.$$