Discussion 6B

CS 70, Summer 2024

1 Inequality Practice

(a) We want to use Markov's Inequality, but recall that Markov's Inequality only works with non-negative random variables. So, we define a new random variable $\tilde{X} = X + 5$, where $\tilde{X}$ is always non-negative, so we can use Markov's on $\tilde{X}$. By linearity of expectation, $\mathrm{E}[\tilde{X}] = -3 + 5 = 2$. So, $\Pr[\tilde{X} \geq 4] \leq 2/4 = 1/2$.

(b) We again use Markov's Inequality. Similarly, define $\tilde{Y} = -Y + 10$, and $\mathrm{E}[\tilde{Y}] = -1 + 10 = 9$. $P[Y \leq -1] = P[-Y \geq 1] = P[-Y + 10 \geq 11] \leq 9/11$.

(c) Let Z_i be the number on the die for the ith roll, for $i = 1, \ldots, 100$. Then, $Z = \sum_{i=1}^{100} Z_i$. By linearity of expectation, $\mathrm{E}[Z] = \sum_{i=1}^{100} \mathrm{E}[Z_i]$.

$$\mathrm{E}[Z_i] = \sum_{j=1}^{6} j \cdot \Pr[Z_i = j] = \sum_{j=1}^{6} j \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j = \frac{1}{6} \cdot 21 = \frac{7}{2}$$

Then, we have $\mathrm{E}[Z] = 100 \cdot (7/2) = 350$.

$$\mathrm{E}[Z_i^2] = \sum_{j=1}^{6} j^2 \cdot \Pr[Z_i = j] = \sum_{j=1}^{6} j^2 \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{j=1}^{6} j^2 = \frac{1}{6} \cdot 91 = \frac{91}{6}$$

Then, we have

$$\mathrm{Var}(Z_i) = \mathrm{E}[Z_i^2] - \mathrm{E}[Z_i]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12},$$

Since the Z_is are independent, and therefore uncorrelated, we can add the $\mathrm{Var}(Z_i)$s to get $\mathrm{Var}(Z) = 100 \times (35/12)$.

Finally, we note that we can upper bound $\Pr[|Z - 350| > 50]$ with $\Pr[|Z - 350| \geq 50]$.

Putting it all together, we use Chebyshev's to get

$$\Pr[|Z - 350| > 50] < \Pr[|Z - 350| \geq 50] \leq \frac{100 \times (35/12)}{50^2} = \frac{7}{60}.$$

2 Crazy High Moments

(a) To see this, we first fully expand the sum,

$$(X_1 + X_2 + \cdots + X_n)^4 = \sum_{i=1}^{n} X_i^4 + \binom{4}{2} \sum_{i<j} X_i^2 X_j^2 + \binom{4}{1} \sum_{i \neq j} X_i X_j^3 + 2 \cdot \binom{4}{2} \sum_{\substack{i,j,k \text{ distinct} \\ j<k}} X_i^2 X_j X_k + 4! \cdot \sum_{i<j<k<t} X_i X_j X_k X_t$$

By linearity of expectation, the expectation of this sum is the same as the sum of the expectations of each term. By indepndence, we know $\mathrm{E}[X_i X_j^3] = \mathrm{E}[X_i] \cdot \mathrm{E}[X_j^3] = 0$ because $\mathrm{E}[X_i] = 0$. Similarly, $\mathrm{E}[X_i^2 X_j X_k] = \mathrm{E}[X_i^2] \cdot \mathrm{E}[X_j] \cdot \mathrm{E}[X_k] = 0$ and $\mathrm{E}[X_i X_j X_k X_t] = \mathrm{E}[X_i]\mathrm{E}[X_j]\mathrm{E}[X_k]\mathrm{E}[X_t] = 0$. This proves that

$$\mathrm{E}[(X_1 + X_2 + \cdots + X_n)^4] = \sum_{i=1}^{n} \mathrm{E}[X_i^4] + \binom{4}{2} \cdot \sum_{i<j} \mathrm{E}[X_i^2] \cdot \mathrm{E}[X_j^2].$$

(b) By linearity of expectation, $\mathrm{E}[Z] = \sum_{i=1}^{100} \mathrm{E}[Z_i]$. We know that $\mathrm{E}[(Z - \mathrm{E}[Z])^4] = \mathrm{E}\left[\left(\sum_{i=1}^{100} Z_i - \mathrm{E}[Z_i]\right)^4\right] = \mathrm{E}\left[\left(\sum_{i=1}^{100} X_i\right)^4\right]$.

Because all the X_i's we defined has expectation zero, we can apply (a).

The problem then boils down to calculating $\mathrm{E}[X_i^4]$ and $\mathrm{E}[X_i^2]\cdot\mathrm{E}[X_j^2]$.

$$\mathrm{E}[X_i^4]=\sum_{j=1}^{6}(j-3.5)^4\cdot\Pr[Z_j=j]=\frac{1}{6}\sum_{j=1}^{6}(j-3.5)^4=\frac{707}{48}.$$

For $\mathrm{E}[X_i^2]\cdot\mathrm{E}[X_j^2]$, we reuse the calculation we had for 1(c), $\mathrm{E}[X_i^2]=\mathrm{Var}[Z_i]=\frac{35}{12}$. $\mathrm{E}[X_i^2]\cdot\mathrm{E}[X_j^2]=\left(\frac{35}{12}\right)^2=\frac{1225}{144}$. Thus

$$\begin{aligned}\mathrm{E}[(Z-\mathrm{E}[Z])^4]&=\mathrm{E}\left[\left(\sum_{i=1}^{100}X_i\right)^4\right]\\&=100\cdot\frac{707}{48}+\binom{4}{2}\cdot\binom{100}{2}\cdot\frac{1225}{144}\\&=\frac{1524775}{6}. &&\text{(apply 2(a).)}\end{aligned}$$

(c) We apply Markov's inequality and get,

$$\Pr[Z>400]\le\Pr[(Z-350)^4>50^4]\le\frac{1524775/6}{50^4}\approx 254129.0/6250000\approx 0.04066.$$

Here we used $1524775/6\approx 254129.0$ as in the hint.

The number we got in 1(c) was $7/60\approx 0.116667$. We get a much tighter tail bound by looking at the fourth moment. With higher moments, we will get even better bounds.

3 Estimating μ and σ^2

(a) $\mathbb{E}[\hat{\mu}]=\mathbb{E}\left[\frac{X_1+\cdots+X_n}{n}\right]=\frac{1}{n}\mathbb{E}[X_1+\cdots+X_n]=\frac{1}{n}n\cdot\mathbb{E}[X_i]=\mathbb{E}[X_i]$.

(b) $\mathrm{E}[X_i^2]=\mathrm{Var}[X_i]+\mathrm{E}[X_i]^2=\sigma^2+\mu^2$ and $\mathrm{E}[X_iX_j]=\mu^2$ $(i\neq j)$.

(c) We first write $\hat{\sigma}^2$ using $X_1,X_2,\ldots,X_n$:

$$\begin{aligned}\hat{\sigma}^2&=\frac{(X_1-\hat{\mu})^2+(X_2-\hat{\mu})^2+\ldots+(X_n-\hat{\mu})^2}{n}\\&=\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\mu})^2\\&=\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2-2X_i\hat{\mu}+\hat{\mu}^2\right)\\&=\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2-2X_i\hat{\mu}\right)+\hat{\mu}^2\\&=\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2-2X_i\cdot\frac{1}{n}\sum_{j=1}^{n}X_j\right)+\left(\frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\\&=\frac{1}{n}\sum_{i=1}^{n}\left(\left(1-\frac{2}{n}+\frac{1}{n}\right)X_i^2+\left(-\frac{2}{n}+\frac{1}{n}\right)\sum_{j:j\neq i}X_iX_j\right)\end{aligned}$$

Therefore, by linearity of expectation,

$$\begin{aligned}
\mathrm{E}[\hat{\sigma}^2] &= \frac{1}{n}\sum_{i=1}^{n}\left(\left(1-\frac{2}{n}+\frac{1}{n}\right)(\sigma^2+\mu^2)+\left(-\frac{2}{n}+\frac{1}{n}\right)\sum_{j:j\neq i}\mu^2\right) \\
&= \left(1-\frac{2}{n}+\frac{1}{n}\right)(\sigma^2+\mu^2)+\left(-\frac{2}{n}+\frac{1}{n}\right)\cdot(n-1)\cdot\mu^2 \\
&= \frac{n-1}{n}\cdot(\sigma^2+\mu^2)-\frac{n-1}{n}\cdot\mu^2 \\
&= \frac{n-1}{n}\sigma^2.
\end{aligned}$$

(d) Propose a modified estimation $\hat{\sigma}^2$ which does satisfy the property.

We use $\hat{\sigma}^2_{\text{unbias}} = \frac{(X_1-\hat{\mu})^2+(X_2-\hat{\mu})^2+\cdots+(X_n-\hat{\mu})^2}{n-1}$. Then

$$\mathrm{E}[\hat{\sigma}^2_{\text{unbias}}] = \frac{n}{n-1}\cdot\mathrm{E}[\hat{\sigma}^2] = \sigma^2.$$