Discussion 6C

CS 70, Summer 2024

1 Duelling Meteorologists

(a) Let S be the event that it snows and T be the event that Tom predicts snow.

$$\begin{aligned}\Pr[S|T] &= \frac{\Pr[S\cap T]}{\Pr[T]}\\ &= \frac{\Pr[S]\cdot \Pr[T|S]}{\Pr[S\cap T]+\Pr[\overline{S}\cap T]}\\ &= \frac{\frac{1}{10}\times\frac{7}{10}}{\frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{5}{100}} = \frac{14}{23}\end{aligned}$$

(b)

$$\begin{aligned}\Pr[A] &= \Pr[S\cap T]+\Pr[\overline{S}\cap\overline{T}]\\ &= \frac{1}{10}\times\frac{7}{10}+\frac{9}{10}\times\frac{95}{100} = \frac{37}{40}\end{aligned}$$

(c) Even though Jerry's overall accuracy is lower, it is still possible that she is a better meteorologist if the weather is different.

For example, let's assume that it snows 80% of days in Alaska.

- When it snows, Jerry correctly predicts snow 80% of the time.
- When it doesn't snow, Jerry correctly predicts no snow 100% of the time.

Jerry's overall accuracy turns out to be less than Tom's even though she is better at predicting both categories! The following diagram gives an illustration of the situation. The intuition is that Jerry's error gets penalized more heavily than Tom because it snows more often in Alaska. This is exactly Simpson's paradox.

2 Monty Hall's Revenge

Throughout the solution, we will refer to W as the event that the contestant wins, and $\mathbb{P}_S[W]$ and $\mathbb{P}_N[W]$ as the probabilities of this event happening if the contestant is (S)witching or (N)ot switching, respectively.

(a) $\mathbb{P}_N[W] = 1/n$ since only one out of n initial choices gets us the car. Under the switching strategy two things can happen: Either the first choice hits the car, and so switching (to any of the remaining $n-2$ doors) will inevitably get us the goat, or our first choice picks a goat, leaving one of the remaining $n-2$ doors with the car. This sequence of choices—first choosing from one of n doors, then switching to one of $n-2$ remaining doors—gives us a sample space of size $n(n-2)$. If we divide the number of favorable outcomes by the total number of outcomes, we get

$$\begin{aligned}\mathbb{P}_S[W] &= \left(\underbrace{(n-1)}_{\text{first choice = goat}}\cdot\underbrace{1}_{\text{second choice = car}}\right)\Big/\underbrace{n(n-2)}_{\text{total \# of choices}}\\ &= \frac{n-1}{n(n-2)} = \frac{1}{n}\cdot\frac{n-1}{n-2}\end{aligned}$$

which is larger than $\mathbb{P}_N[W] = 1/n$ (ever so slightly so the larger n becomes, which demonstrates the intuitive fact that Monty's help gets decreasingly helpful the more doors there are), so switching doors is the better strategy.

(b) $\mathbb{P}_N[W] = 1/n$ remains unchanged. The same approach as in part (a) yields the same numerator as before. For the denominator, we need to figure out the size of the sample space for the experiment where we first pick a door at random, then switch. Again, there are n ways of making the first choice. Once Monty reveals $n-2$ other doors, though, there is only one remaining option for us to switch to. Thus the denominator is much smaller:

$$\begin{aligned}\mathbb{P}_S[W] &= \left(\underbrace{(n-1)}_{\text{first choice = goat}} \cdot \underbrace{1}_{\text{second choice = car}} \right) \bigg/ \underbrace{n \cdot 1}_{\text{total \# of choices}} \\ &= \frac{n-1}{n} = 1 - \frac{1}{n}\end{aligned}$$

so switching is again the better strategy.

(c) Now $\mathbb{P}_N[W] = k/n$ since k doors hide a car. Reasoning about sample spaces in the same way we did in part (b) gives us a way to compute the denominator of $\mathbb{P}_S[W]$. However, now the numerator (number of favorable outcomes in the case where we switch) changes too:

$$\begin{aligned}\mathbb{P}_S[W] &= \left(\underbrace{k}_{\text{first choice = car}} \cdot \underbrace{k-1}_{\text{second choice = car}} + \underbrace{(n-k)}_{\text{first choice = goat}} \cdot \underbrace{k}_{\text{second choice = car}} \right) \bigg/ \underbrace{n(n-j-1)}_{\text{total \# of choices}} \\ &= \frac{k(n-1)}{n(n-j-1)} = \frac{k}{n} \cdot \frac{n-1}{n-j-1}.\end{aligned}$$

From here we see that $\mathbb{P}_S[W]/\mathbb{P}_N[W] = \frac{n-1}{n-j-1}$, which is maximal if $j = n-k-1$. In other words, if Monty reveals all but one goat (which he does in the original show where $n = 3, k = 1$ and $j = 1 = n-k-1$), then the contestant can increase their chances of winning by a factor of $\frac{n-1}{k}$ (which is a factor of 2 in the original show). In particular, the largest relative advantage of switching is achieved when $k = 1$.