Discussion 6D

CS 70, Summer 2024

1 Random Walk

(a) $\Pr[X_1 = 3] = 0.2 * 0.9 + 0.3 * 0.4 + 0.1 * 0.8 + 0.3 * 0.7 = 0.59$.

(b) $\Pr[X_0 = 2 \wedge X_1 = 4 \wedge X_2 = 3 \wedge X_3 = 3 \wedge X_4 = 2 \wedge X_5 = 4] = 0.3 * 0.6 * 0.7 * 0.8 * 0.2 * 0.6$.

(c) $(0, 0, 0, 0, 1)$. If we run this Markov chain for long enough from any initial distribution, we will always eventually end up trapped at 5. So this is the unique stationary distribution.

(d) Let stationary distribution be $(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5))$. We get

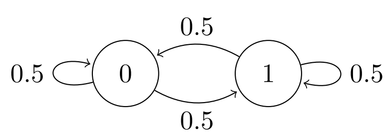
$$\begin{aligned} 0.2 \cdot \pi(2) &= \pi(1) \\ 0.2 \cdot \pi(3) &= \pi(2) \\ 0.9 \cdot \pi(1) + 0.2 \cdot \pi(2) + 0.8 \cdot \pi(3) + \pi(4) &= \pi(3) \\ 0.1 \cdot \pi(1) + 0.6 \cdot \pi(2) &= \pi(4) \end{aligned}$$

2 Build Your Markov Chain

For each of the following random processes, propose a finite-size Markov chain to describe it. You should both draw the diagram, write down the transition matrix P for the Markov chain you build, and the starting distribution π_0.

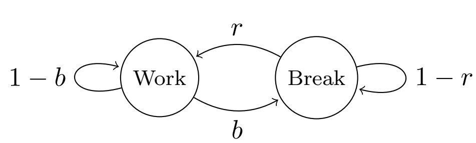
In the following, H stands for "head" and T stands for "tail".

(a) We use the states to represent the value of Z_i.



$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \pi_0 = (1, 0).$$

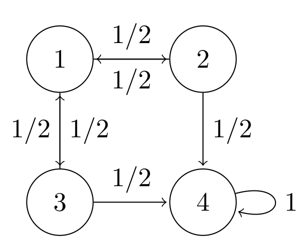
(b) The Markov chain is as follows:



Suppose we label "Work" as node 1 and "Break" as node 2.

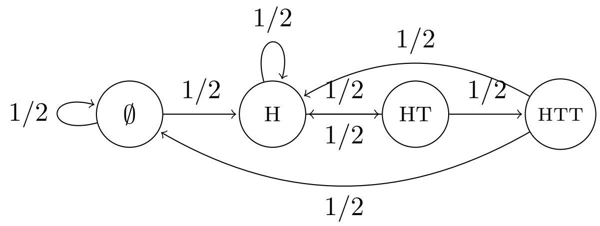
$$P = \begin{pmatrix} 1-b & b \\ r & 1-r \end{pmatrix}, \pi_0 = (1/2, 1/2).$$

(c) The Markov chain is as follows:



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \pi_0 = (1, 0, 0, 0).$$

(d) We have four different states: 1. The last flips matches nothing; 2. The last one flip matches H; 3. THe last one flip matches HT; 4. The last one flip matches HTT.



The value of Y_i is 0 when we are in states $\emptyset$, H, HT and 1 when we are at HTT.

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}, \pi_0 = (1, 0, 0, 0).$$

3 Markov Chain Terminology

(a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one of a and b is 0.

(b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2, 4, 6, ...\} = 2.$$

This is because if we start at a state X then we can get back to it after taking an even number of steps only (2, 4, 6, 8, etc.), not by taking an odd number of steps (1, 3, 5, 7, etc.). Thus, the chain is periodic with period 2.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1, 2, 3, ...\} = 1.$$

Thus, the chain is aperiodic. Notice that the self-loops allow us to stay at the same node, thereby letting us get to any other node in an odd *or* even number of steps.

(d)

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e)

$$\pi(0) = (1-b)\pi(0) + a\pi(1),$$
$$\pi(1) = b\pi(0) + (1-a)\pi(1).$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

$$\pi = \frac{1}{a+b}\begin{bmatrix} a & b \end{bmatrix}.$$