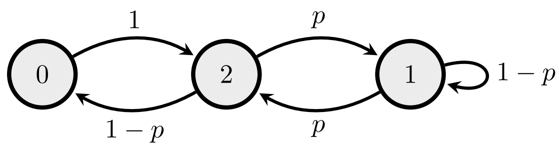
Discussion 7A

CS 70, Summer 2024

1 Wet Professor

(a) Let the state space be the number of umbrellas with the professor: $S = \{0, 1, 2\}$. Then the Markov chain is given below.



(b) The chain is irreducible because the loop $0 \to 2 \to 1 \to 2 \to 0$ visits every state. So it is possible to get from any state to any other state.

The chain is irrreducible, so every state shares the same period. The state 1 has a self-loop, so its period is $d(1) = \gcd\{1, 2, 3, \ldots\} = 1$. So all states have period 1, and thus the chain is aperiodic.

(c) This is a question about the long run, so we need to find the stationary distribution π of the Markov chain. Let $q = 1 - p$. By the balance equations,

$$\begin{aligned}\pi(0) &= q\pi(2)\\ \pi(1) &= q\pi(1) + p\pi(2)\\ \pi(2) &= \pi(0) + p\pi(1).\end{aligned}$$

The second equation yields $\pi(1) = \pi(2)$. The third equation is redundant.

So we have that $\pi(0) = q\pi(2)$ and $\pi(1) = \pi(0)$. Since π is a distribution, it must sum to one. Therefore

$$\pi(0) + \pi(1) + \pi(2) = q\pi(2) + \pi(2) + \pi(2) = 1 \implies \pi(2) = \frac{1}{2+q}.$$

Therefore

$$\pi(0) = \frac{q}{q+2} \qquad \pi(1) = \frac{1}{2+q}.$$

The probability that the professor gets wet is

$$\mathrm{P}(X_n = 0 \text{ and it rains}) \to \pi(0) \cdot p = \frac{pq}{2+q}.$$

(d) The professor only gets wet when she has no umbrellas and it rains.

Let N be the number of walks the professor takes with no umbrellas until it rains. Then $N \sim \text{Geometric}(p)$, since it rains on each such walk with probability p, independently of all other walks.

Let $T_1, \ldots, T_N$ be the number of walks the professor takes between each walk with no umbrellas. That is, T_1 is the number of walks it takes for the professor to return to state 0 after leaving state 0 for the first time, T_2 is the number of walks it takes for the professor to return to state 0 after leaving state 0 for the second time, and so on.

Then for W the number of walks until the professor gets wet,

$$W = T_1 + \ldots + T_{N-1} + 1.$$

The $+1$ is for the N^{th} walk, on which she gets wet. Then

$$\mathrm{E}[W \mid N = n] = \mathrm{E}[T_1 + \ldots + T_{n-1} + 1] = (n-1)\mathrm{E}[T_1] + 1 = (n-1)\frac{2+q}{q} + 1,$$

where $\mathrm{E}[T_1] = 1/\pi(0)$. Then

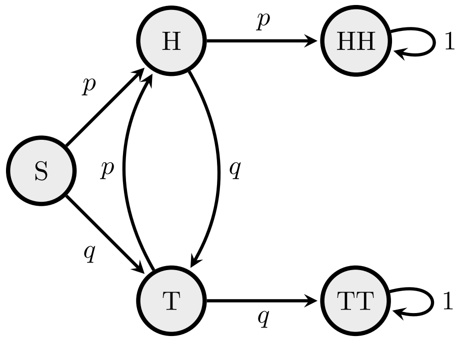
$$\mathrm{E}[W \mid N] = (N-1)\frac{2+q}{q} + 1,$$

so

$$\mathrm{E}[W] = \mathrm{E}[\mathrm{E}[W \mid N]] = \mathrm{E}\left[(N-1)\frac{2+q}{q}\right] + 1 = \frac{2+q}{q}\left(\mathrm{E}[N] - 1\right) + 1 = \frac{2+q}{q}\left(\frac{1}{p} - 1\right) + 1 = \frac{2+q}{p} + 1 = \frac{3}{p}.$$

2 Double Toss

(a) This game can be represented with the following Markov chain. Let $q = 1 - p$.



Note that the transitions at states HH and TT don't matter.

Based on this Markov chain, we have the following first-step equations for P(Amari wins) = P(HH before TT).

Let $\alpha(s) = \mathrm{P}(\text{HH before TT} \mid X_0 = s)$ be the probability that the coin tosses two consecutive heads before two consecutive tails given that we are already in state $s \in \{\text{S}, \text{H}, \text{T}, \text{HH}, \text{TT}\}$. Then we have the following first-step equations.

$$\begin{aligned}
\alpha(\text{S}) &= p\alpha(\text{H}) + q\alpha(\text{T}) \\
\alpha(\text{H}) &= p\alpha(\text{HH}) + q\alpha(\text{T}) \\
\alpha(\text{T}) &= p\alpha(\text{H}) + q\alpha(\text{TT}) \\
\alpha(\text{HH}) &= 1 \\
\alpha(\text{TT}) &= 0.
\end{aligned}$$

Simplifying these equations yields $\alpha(\text{T}) = p\alpha(\text{H})$ and $\alpha(\text{H}) = p + q\alpha(\text{T})$. Therefore

$$\alpha(\text{H}) = p + q\alpha(\text{T}) = p + qp\alpha(\text{H}) \implies \alpha(\text{H}) = \frac{p}{1-qp} \implies \alpha(\text{T}) = \frac{p^2}{1-qp}.$$

Therefore

$$\mathrm{P}(\text{Amari wins}) = \alpha(\text{S}) = p\alpha(\text{H}) + q\alpha(\text{T}) = \frac{p^2}{1-qp} + \frac{qp^2}{1-qp} = \frac{p^2(1+q)}{1-qp}.$$

(b) Let D be the number of tosses until the game ends and let $\beta(s) = \mathrm{E}[D \mid X_0 = s]$ be the number of tosses it takes for the game to end when starting at state $s \in \{\text{S}, \text{H}, \text{T}, \text{HH}, \text{TT}\}$. The first-step equations are as follows.

$$\begin{aligned}
\beta(\text{S}) &= p(1 + \beta(\text{H})) + q(1 + \beta(\text{T})) \\
\beta(\text{H}) &= p(1 + \beta(\text{HH})) + q(1 + \beta(\text{T})) \\
\beta(\text{T}) &= p(1 + \beta(\text{H})) + q(1 + \beta(\text{TT})) \\
\beta(\text{HH}) &= 0 \\
\beta(\text{TT}) &= 0.
\end{aligned}$$

These simplify to

$$\begin{aligned}
\beta(\text{S}) &= 1 + p\beta(\text{H}) + q\beta(\text{T}) \\
\beta(\text{H}) &= 1 + q\beta(\text{T}) \\
\beta(\text{T}) &= 1 + p\beta(\text{H}).
\end{aligned}$$

Combining the last two equations yields

$$\beta(\text{T}) = 1 + p\,(1 + q\beta(\text{T})) = 1 + p + pq\beta(\text{T}) \implies \beta(\text{T}) = \frac{1+p}{1-pq} \implies \beta(\text{H}) = 1 + q\frac{1+p}{1-pq} = \frac{1+q}{1-pq}.$$

Then

$$\mathrm{E}[D \mid X_0 = \text{S}] = \beta(\text{S}) = 1 + p\frac{1+q}{1-pq} + q\frac{1+p}{1-pq} = \frac{2+pq}{1-pq}.$$

(c) Let R be the number of heads until the game ends and let $\gamma(s) = \mathrm{E}[R \mid S = s]$ be the number of heads it takes for the game to end when starting in state s. The first-step equations are as follows.

$$\begin{aligned}
\gamma(\mathrm{S}) &= p(1 + \gamma(\mathrm{H}) + q\gamma(\mathrm{T}) \\
\gamma(\mathrm{H}) &= p(1 + \gamma(\mathrm{HH})) + q\gamma(\mathrm{T}) \\
\gamma(\mathrm{T}) &= p(1 + \gamma(\mathrm{H})) + q\gamma(\mathrm{TT}) \\
\gamma(\mathrm{HH}) &= 0 \\
\gamma(\mathrm{TT}) &= 0.
\end{aligned}$$

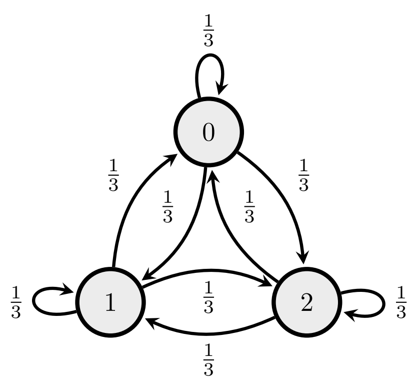
Solving this yields

$$\gamma(\mathrm{S}) = p + p\frac{p + pq}{1 - pq} + q\frac{p(1 + p)}{1 - pq} = \frac{p(2 + pq)}{1 - pq} = p\beta(\mathrm{S}).$$

This is exactly the answer we expected, since each toss is heads with probability p. More formally, we could have gotten this same answer by noticing that $R = I_1 + \ldots + I_D$, where $I_j = 1$ if the j^{th} toss is heads. Solving for $\mathrm{E}[R]$ by conditioning yields $\mathrm{E}[R] = p\mathrm{E}[D] = p\beta(\mathrm{S})$.

3 Die Sum

(a) The Markov chain has three states representing the three remainders modulo 3.



(b) The chain is irreducible because the loop $0 \to 1 \to 2 \to 0$ visits every state. It is aperiodic because there are self-loops. In particular, $d(0) = \gcd\{1, 2, \ldots\} = 1$.

(c) By symmetry, the stationary distribution is uniform over the chain. In particular, the three states are identical, so no state can have a higher probability of occurring in the long run than any other state. So the stationary distribution is

$$\pi = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

(d) This is $1/\pi(0) = 3$.