Discussion 7B

CS 70, Summer 2024

1 Random Variables with Density

(a) Since f is a density, it must integrate to 1. Therefore

$$1 = \int_{-1}^{1} c(1-x^2)\,\mathrm{d}x = c\left(\left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1}\right) = c\left(\left(1-\frac{1}{3}\right) - \left(-1-\left(-\frac{1}{3}\right)\right)\right) = c\left(\frac{4}{3}\right).$$

Solving for c yields

$$c = \frac{3}{4}.$$

(b) By definition, $F(x) = \mathrm{P}(X \le x)$. For $x \le -1$, $F(x) = 0$, and for $x \ge 1$, $F(x) = 1$.

For $-1 < x < 1$, we have that

$$F(x) = \mathrm{P}(X \le x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)\,\mathrm{d}t = \frac{3}{4}\left(t - \frac{1}{3}t^3\right)\Big|_{-1}^{x} = \frac{3}{4}\left(x - \frac{1}{3}x^3 + \frac{2}{3}\right)$$

Therefore the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x \le -1, \\ \frac{3}{4}\left(x - \frac{1}{3}x^3 + \frac{2}{3}\right) & -1 < x < 1, \\ 1 & x \ge 1. \end{cases}$$

(c) This is $\mathrm{P}(-0.1 < X < 0.1) = F(0.1) - F(-0.1)$. By the symmetry of the density function, this simplifies to $1 - 2F(-0.1)$.

(d) By the symmetry of the density function, this is 0.

We can also compute it. By the definition of expectation,

$$\mathrm{E}[X] = \int_{-1}^{1} x \cdot \frac{3}{4}(1-x^2)\,\mathrm{d}x = \frac{3}{4}\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_{-1}^{1} = \frac{3}{4}\left(\left(\frac{1}{2}-\frac{1}{4}\right) - \left(\frac{1}{2}-\frac{1}{4}\right)\right) = 0.$$

(e) By the law of the unconscious statistician,

$$\mathrm{E}[X^2] = \int_{-1}^{1} x^2 \cdot \frac{3}{4}(1-x^2)\,\mathrm{d}x = \frac{3}{4}\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_{-1}^{1} = \frac{3}{4}\left(\left(\frac{1}{3}-\frac{1}{5}\right) - \left(-\frac{1}{3}-\left(-\frac{1}{5}\right)\right)\right) = \frac{1}{5}.$$

Therefore

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \frac{1}{5} - 0^2 = \frac{1}{5}.$$

2 Exponential Minima

As we have seen in the lecture and homeworks, the minimum of a sample is large if all the elements of the sample are large. So we find the survival function of V.

$$\begin{aligned} \mathrm{P}(V > v) &= \mathrm{P}(X_1 > v, \ldots, X_n > v) \\ &= \mathrm{P}(X_1 > v) \cdot \ldots \cdot \mathrm{P}(X_n > v) && \text{(independence)} \\ &= e^{-\lambda_1 v} \cdot \ldots \cdot e^{-\lambda_n v} && \text{(exponential survival function)} \\ &= e^{-\left(\sum_{i=1}^{n} \lambda_i\right) v}. \end{aligned}$$

Note that this is the same as the survival function of an exponential random variable with rate parameter $\lambda_1 + \ldots + \lambda_n$. So that must be the distribution of V:

$$V \sim \text{Exponential}\left(\sum_{i=1}^{n} \lambda_i\right).$$

3 Competing Uniforms

(a) By drawing the region over which X and Y are defined and the region over which $X > Y$, we can see that this event covers one quarter of the total region over which X and Y are defined.

Since X and Y are uniform and independent, we thus have that $\mathrm{P}(X > Y) = 1/4$.

(b) If we knew the value of X, we would be able to find the probability that $X > Y$. That's our cue to partition.

$$\begin{aligned}
\mathrm{P}(X > Y) &= \sum_{k=0}^{n} \mathrm{P}(X > Y \mid X = k)\mathrm{P}(X = k) \\
&= \sum_{k=0}^{n} \mathrm{P}(Y < k)\mathrm{P}(X = k) && \text{(independence)} \\
&= \sum_{k=0}^{n} \frac{k}{n}\frac{1}{n+1} \\
&= \frac{1}{n}\sum_{k=0}^{n} k \cdot \frac{1}{n+1} \\
&= \frac{1}{n}\sum_{k=0}^{n} k\mathrm{P}(X = k) \\
&= \frac{1}{n}\mathrm{E}[X] \\
&= \frac{1}{n} \cdot \frac{n}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Another way to see this is by conditioning. Observe that $\mathrm{P}(X > Y \mid X = k) = k/n$, so

$$\mathrm{P}(X > Y \mid X) = \frac{X}{n}.$$

By the law of total probability/expectation,

$$\mathrm{P}(X > Y) = \mathrm{E}[\mathrm{P}(X > Y \mid X)] = \mathrm{E}\left[\frac{X}{n}\right] = \frac{1}{2}.$$

4 Functions of Uniforms

(a) The possible values of U^2 are $(0, 1)$. We'll find the cumulative distribution function of U^2. For $u \in (0, 1)$,

$$\begin{aligned}
F_{U^2}(u) &= \mathrm{P}(U^2 \leq u) \\
&= \mathrm{P}(U \leq \sqrt{u}) && (U \geq 0, u \geq 0) \\
&= \sqrt{u}.
\end{aligned}$$

Therefore the density of U^2, for $u \in (0, 1)$, is given by

$$f_{U^2}(u) = \frac{\mathrm{d}}{\mathrm{d}u}\sqrt{u} = \frac{1}{2\sqrt{u}}, \qquad 0 < u < 1.$$

(b) The possible values of $|V|$ are $(0, 1)$. Again, we find the cumulative distribution function. For $v \in (0, 1)$,

$$\begin{aligned}
F_{|V|}(v) &= \mathrm{P}(|V| \leq v) \\
&= \mathrm{P}(-v \leq V \leq v) \\
&= F_V(v) - F_V(-v) \\
&= \frac{v+1}{2} - \frac{-v+1}{2} \\
&= v.
\end{aligned}$$

This is the cumulative distribution function of the uniform $(0, 1)$ random variable. So $|V| \sim \mathrm{Uniform}(0, 1)$.

(c) The possible values of $1/U$ are $(1, \infty)$. We'll find the cumulative distribution function of $1/U$. For $u \in (1, \infty)$,

$$\begin{aligned} F_{1/U}(u) &= \mathrm{P}(1/U \leq u) \\ &= \mathrm{P}(U \geq 1/u) && (U > 0, u > 0) \\ &= 1 - \frac{1}{u}. \end{aligned}$$

Therefore the density of $1/U$ is given by

$$f_{1/U}(u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(1 - \frac{1}{u}\right) = \frac{1}{u^2}, \qquad u > 1.$$