Final Review

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2026

Final Exam – Student Request Topics

- Graph/edge coloring

  Vertex and edge coloring problems are good sources of problems to check your intuition of basic graph properties. Let's look at some problems on the board, including:

  - How is vertex coloring related to degree? Can you think of a graph with max degree 2 that requires 3 colors? Can you think of a graph with max degree 100 that only needs 2 colors? How many colors does K_n require?
  - Combining graphs: Let G_1 and G_2 be two graphs that can each be colored with 2 colors. If you connect them with a single edge to create a new graph G, how many colors do you need for G? What if you connect with *two* edges?
  - Edge coloring: For examples above, how many edge colors are needed?

Final Exam – Student Request Topics Continued

- Order statistics / the CDF method (e.g. Fa24 14.1–14.4, Fa23 15): going from max/min of independent uniforms to a CDF, then differentiating to a PDF.

  Let's work through 14.1 and 14.2 together (14.3 and 14.4 follow easily):

  X, Y are independent and uniform on $[0, r]$, and let $Z = \max(X, Y)$ – what are the CDF and PDF of Z?

- First-step equations, specifically which system to write (e.g. Fa24 20.1 and 21.2, Fa23 17): when it's $\beta = 1 + P\beta$ vs $\alpha = P\alpha$ with no +1 vs $\pi = \pi P$, and where the +1 comes from.

  This will be in the following review slides...

- Counting solutions to $ax \equiv b \pmod{pq}$ (e.g. Fa24 6.4): counting invertible a, and how the solution count changes when gcd(a, pq) isn't 1.

  Let's work this out when $\gcd(a, pq) = 1$ and when $\gcd(a, pq) = p$.

- The abstract Δ_i polynomials and Berlekamp-Welch (e.g. Fa24 7.3).

  These are Lagrange polynomials for interpolation. Let's look at $\Delta_i(i)$ and $\Delta_i(j)$ (for j in set of x values).

Lecture 13 Summary – Counting II

Combinatorial Proofs: Identity from counting same set in two ways

- Basic combinations: $\binom{n}{k} = \binom{n}{n-k}$ ways to include k = ways to exclude $n-k$
- Binomial Coefficients and Binomial Theorem
  - Expansion of monomial power is ways of choosing *a* factors
- Pascal's Triangle and Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.
  - LHS: Count as number of subsets of $n+1$ items size k.
  - RHS: $\binom{n}{k-1}$ counts subsets of $n+1$ items with first item.
    - $\binom{n}{k}$ counts subsets of $n+1$ items without first item.
    - Disjoint – so add!
- Hockey Stick Identity
  - Count as subsets, and count as subsets with given smallest element

Inclusion/Exclusion: Two sets of objects

- Add number of each subtract intersection of sets
- Sum Rule: If intersection is empty, nothing to subtract!

Inclusion/Exclusion: General

- Alternate adding/subtracting: Add 1-way, subtract 2-way, add 3-way, ...

Concept Check

What "two ways of counting" give a combinatorial proof that $k\binom{n}{k} = n\binom{n-1}{k-1}$?

For three sets A, B, and C, how many intersection subsets do you subtract and add from $|A|+|B|+|C|$ to get $|A \cup B \cup C|$?

Lecture 14 Summary – Countability

Sizes of sets

- Comparing sizes of two sets – bijections
- Infinite sets too!

Countably infinite sets

- Counting numbers ($\mathbb{N}$) are countable (surprise!)
- Integers are countable
- Rationals are countable
- The set of all finite-length binary strings is countable

Uncountable sets

- The set of reals is uncountable
- Diagonalization as a proof technique
- The set of subsets of $\mathbb{N}$ is uncountable

Bottom line: Infinity is weird. And cool.

Concept Check

Countable or uncountable?

- The set of all finite length strings of English letters.
- The set of all subsets of integers.
- The set of all finite length decimal representations of real numbers.

True or false: Every surjective (onto) mapping from integers to even integers is a bijection.

Lecture 15 Summary – Computability

The problem with self-reference
- Liar's Paradox, Russell's Paradox, ...

Uncomputable functions
- Programs as data
- The halting problem
  - Proof of undecidability
  - Diagonalization view of the proof

Some History
- Alan Turing
- David Hilbert

Reductions: Solving one problem with another
- Undecidability of basic program behavior
- HALT reduction to "prints hello world"

Some additional uncomputable functions

Gödel's Incompleteness Theorem

Concept Check

- What basic proof technique is at the heart of the undecidability proof of the halting problem?
- True or false: We can write a program analyzer that takes a program as input and always correctly determines if the program performs a division by zero when it is run.
- True or false: We can write a program analyzer that takes a program as input and always correctly determines if the program follows style guidelines for variable names.

Lecture 16: Summary – Discrete Probability Basics

1. Random Experiment
2. Probability Space: Ω; $\quad \mathbb{P}[\omega] \in [0,1]$; $\quad \sum_{\omega} \mathbb{P}[\omega] = 1$
3. Uniform Probability Space: $\mathbb{P}[\omega] = 1/|\Omega|$, for all $\omega \in \Omega$
4. Event, "subset of outcomes": $A \subseteq \Omega$, and $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$

   $\Rightarrow$ On a uniform space (common!), just counting $|A|$ since $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$
5. Examples!

Concept Check

True or false:

- For events A and B, it's possible that $\mathbb{P}[A] + \mathbb{P}[B] > 1$.

- The sample space Ω always has finite size.

For the experiment of rolling two six-sided dice, how many points are in the sample space?

Lecture 17 Summary – Combinations of Events

Topics:

- Conditional Probability and Bayesian Inference

  $\mathbb{P}[A|B] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]}$

- Total Probability

  $\mathbb{P}[A] = \mathbb{P}[A|B]\cdot\mathbb{P}[B] + \mathbb{P}[A|\overline{B}]\cdot\mathbb{P}[\overline{B}]$

- Bayes' Rule

  $\mathbb{P}[A|B] = \frac{\mathbb{P}[B|A]\cdot\mathbb{P}[A]}{\mathbb{P}[B]}$

- Independent Events and Event Intersection

  Independent A and B $\iff$ $\mathbb{P}[A\cap B] = \mathbb{P}[A]\cdot\mathbb{P}[B]$

- Intersection for Non-Independent Events (Product Rule)

  $\mathbb{P}[A\cap B\cap C] = \mathbb{P}[A]\cdot\mathbb{P}[B|A]\cdot\mathbb{P}[C|A\cap B]$

- Event Union and Inclusion-Exclusion

  $\mathbb{P}[A\cup B] = \mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A\cap B]$

Note: Formulas are only simplest forms, as reminders

Concept Check

True or false?

- If events $B_1, \ldots, B_k$ partition the sample space, then $\mathbb{P}[A \cap B_1] + \mathbb{P}[A \cap B_2] + \cdots + \mathbb{P}[A \cap B_k] = 1$.

- If events $B_1, \ldots, B_k$ partition the sample space, then $\mathbb{P}[B_1 | A] + \mathbb{P}[B_2 | A] + \cdots + \mathbb{P}[B_k | A] = 1$.

- For events A and B, $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$.

Under what condition(s) is $\mathbb{P}[A|B] = \mathbb{P}[B|A]$?

Lecture 18 Summary – Applications

Considered three real applications – all really just “balls and bins!”

- Analyzing hash functions

  For: data structures, file identification/integrity, cryptography, blockchain, ...

  $\mathbb{P}[\textit{collision}] = p \approx e^{-m^2/2n}$

  Given any two of p, m, and n, can solve for the other

- Data coverage by random sampling

  Exploring randomly – how long to see everything?

  Coupon collector: Get all with probability $\geq 1/2$ with $n \ln(2n)$ samples

- Load balancing

  Spreading out computational work or resources evenly

  When $m = n$, probability is $\leq 1/2$ that worst-case bin has $> \frac{\ln n}{\ln \ln n}$ balls

Concept Check

For large n, if we draw m samples with replacement, approximately what should m be for there to be a 50-50 chance of a collision?

True or False: Every child would love to collect cards featuring Turing Award winners.

Lecture 19 Summary – Random Variables

- A random variable X is a function $X : \Omega \to \mathbb{R}$
- Preimages $X^{-1}(a)$ are just events

  $\mathbb{P}[X = a] := \mathbb{P}[X^{-1}(a)] = \mathbb{P}[\{\omega \mid X(\omega) = a\}]$
- Distribution of X is the set of pairs: $\{(a, \mathbb{P}[X = a]), a \in \mathscr{A}\}$.
- Expected value: $\mathbb{E}[X] := \sum_a a \cdot \mathbb{P}[X = a]$
- Common distributions:

  Bernoulli (success in one trial): $X \sim \text{Ber}(p)$

  Binomial (success in n trials): $X \sim \text{Bin}(n, p)$
- Multiple random variables:

  Joint distributions and marginal distributions

  Independence
- Linearity of expectation: $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$
- Examples: Rolling dice, flipping coins, fixed points of permutation, ...

Concept Check

True or False:

- A random variable can have values that are pairs of integers.
- Events $X = 1$ and $X = 2$ with non-zero probabilities are independent.

The Binomial distribution is the sum of independent random variables with what distribution?

If $X \sim \text{Binomial}(10, p)$ and $Y \sim \text{Binomial}(10, q)$, and $Z = X + Y$, are X and Z independent? What is $\mathbb{E}[X + Z]$?

Lecture 20 Summary

Conditional expectation

Basic properties: $\mathbb{E}[X \mid Y = y] = \sum_x x \cdot \mathbb{P}[X = x \mid Y = y]$

Law of Total Expectation: $\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y] \cdot \mathbb{P}[Y = y]$

Functions of a random variable

Sums and products

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$; *when independent,* $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

Variance

Definitions and meaning

$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = (\mathbb{E}[X])^2 - \mathbb{E}[X^2]$

Variance of common distributions: Bernoulli, Binomial, ...

Properties of variance

$\mathrm{Var}(X + c) = \mathrm{Var}(x)$; $\mathrm{Var}(c \cdot X) = c^2 \cdot \mathrm{Var}(X)$; *Ind*: $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Covariance

Definition and meaning: $\mathrm{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

Positively and negatively correlated random variables

Correlation: Scaled covariance: $\mathrm{Corr}(X, Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$

Concept Check

True or False: $\text{Var}(cX) = c\text{Var}(X)$.

For random variables X and Y, what condition guarantees that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

For random variables X and Y, what is the maximum possible value of $\text{cov}(X, Y)$?

For random variables X and Y, what is the maximum possible value of $\text{Corr}(X, Y)$?

Lecture 21 Summary

Added two more common distributions to our toolbox!

Geometric distribution: $X \sim \text{Geometric}(p)$

- $\mathbb{P}[X = k] = (1-p)^{k-1}p \quad (\text{for } i \geq 1) \quad ; \quad \mathbb{E}[X] = \frac{1}{p} \quad ; \quad \text{Var}(X) = \frac{1-p}{p^2}$
- Memoryless distribution
- Used to analyze expected number of steps in coupon collecting

Poisson distribution: $X \sim \text{Poisson}(\lambda)$

- $\mathbb{P}[X = k] = \frac{k}{k!}e^{-\lambda} \quad (\text{for } i \geq 0) \quad ; \quad \mathbb{E}[X] = \lambda \quad ; \quad \text{Var}(X) = \lambda$
- Events happen at a given expected rate
- Limiting distribution of increasingly fine-grained Binomial
- Sum of two Poisson is Poisson

Concept Check

Which distribution is memoryless?

Which distribution is useful to find the expected value of the coupon collector's problem?

The sum of two random variables with Poisson distributions with different parameters has what distribution?

The sum of two random variables with binomial distributions with different parameters has what distribution?

The sum of two random variables with binomial distributions with the same parameters has what distribution?

Lecture 22 Summary

Markov's Inequality: $\mathbb{P}[X \geq c] \leq \frac{\mathbb{E}[X]}{c}$

- X must be a non-negative random variable (i.e., $\mathbb{P}[X < 0] = 0$)
- All you need to know about X is $\mathbb{E}[X]$
- Best bound possible if all you know is $\mathbb{E}[X]$, but often not good enough

Chebyshev's Inequality: $\mathbb{P}[|X - \mu| \geq c] \leq \frac{\text{Var}(X)}{c^2}$

- Uses Var(X) as indication of "spread" of distribution
- Best possible bound if all you know is $\mathbb{E}[X]$ and Var(X)
- Gives good results for estimating probabilities and polling
- Confidence intervals – estimating $\mathbb{E}[X]$ with average of n samples A_n:

  $n = \frac{Var(X)}{\delta\varepsilon^2}$ guarantees $\mathbb{P}[|A_n - \mu| \geq \varepsilon] \leq \delta$

  Estimating probability p (polling): Use $n = \frac{1}{4\delta\varepsilon^2}$

(Weak) Law of Large Numbers $\lim_{n\to\infty} \mathbb{P}[|A_n - \mu| \geq \varepsilon] = 0$

Concept Check

What bound for the tail of a distribution of random variable X uses only expected value of X?

What condition must be met in order to use Markov's inequality?

True or False: There are distributions for which Chebyshev's inequality is tight (i.e., gives the best possible bound).

Lecture 23+24 Summary – Continuous (Two Days)

Two days on continuous probability

Continuous Random Variables
- Idea and Probability Density
- Cumulative Density Function
- Examples

Important Distribution 1: Exponential

Probability Concepts in a Continuous Setting
- Expectation and Variance
- Joint and Marginal Distributions
- Conditional distribution, Independence, and Total Probability

Important Distribution 2: Normal (Gaussian)
- Definition
- Shifting and Scaling
- Standard deviation bands
- Central Limit Theorem

Concept Check

Which continuous distribution is memoryless?

How do you compute the PDF from the CDF?

True or False: If X is distributed as a standard normal distribution, then the probability that $-1 \leq X \leq 1$ is approximately 95%.

If $X \sim N(45, 10)$ and $\mathbb{P}[X \leq 35] = 0.159$, what is $\mathbb{P}[X \geq 55]$?

Lecture 25 Summary – Markov Chains I

A Markov Chain is a series of random variables, such that

- We know the distribution of X_0, denoted as π_0.
- We know the transition probability matrix P.
- $P(X_{\text{next}} = j | X_{\text{prev}} = i) = P(i,j)$
- The distribution of kth random variable in the chain $\pi_k = \pi_0 P^k$

"First step analysis" problems

- Computations are often easier than other "lower level" techniques.
- Examples:
  - Finding the expected time to reach a given state (getting two tails in a row).
  - Finding the probability of reaching one state before another (making \$100 before running out of money).

Concept Check

Row or Column: The sum of the values in a ________ of the transition matrix must be 1.

True or False: To determine the probability distribution of state X_{t+1} you must know the history of all previous states $(X_1, \ldots, X_t)$.

If "next step equations" look like $f_1(x) = 1 + c_1 f_1(x) + c_2 f_2(x) + ...$, are the equations for "hitting time" or the "A before B" problem?

Lecture 26 Summary – Markov Chains II

The Stationary Distribution – two central questions:

1. Is there a unique stationary distribution?

   Yes if and only if the Markov Chain is irreducible

2. Do state probabilities converge to the stationary from any initial distribution?

   If and only if the Markov Chain is irreducible and aperiodic (period 1)

   Known as the "Fundamental Theorem of Markov Chains"

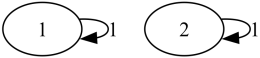
Application: Predicting Text

Idea: Use Markov Chain to predict (or generate!) text

Context is state for Markov chain, and transitions are for next symbol/token

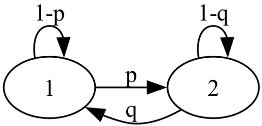
Concept Check

Does the following Markov chain have a stationary distribution?



Does it have a unique stationary distribution?

Does the following Markov chain converge to a unique stationary distribution for every initial distribution?



Fin

Good luck tomorrow!