CS 70 Discrete Mathematics and Probability Theory
Spring 2025 Rao

HW 01

1 Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t.$$

(b) Compute the double integral

$$\iint_R 2x + y\,\mathrm{d}A,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

Solution:

(a) Let $I = \int \sin(t)e^{-t}\,\mathrm{d}t$.
Use integration by parts, with $u = \sin(t)$ and $\mathrm{d}v = e^{-t}$.
This means $\mathrm{d}u = \cos(t)$ and $v = -e^{-t}$.

$$\begin{aligned} I = \int \sin(t)e^{-t}\,\mathrm{d}t &= uv - \int v\cdot \mathrm{d}u \\ &= -\sin(t)e^{-t} + \int e^{-t}\cos(t)\,\mathrm{d}t \end{aligned}$$

Use integration by parts again on $\int e^{-t}\cos(t)\,\mathrm{d}t$, with $u = \cos(t)$ and $\mathrm{d}v = e^{-t}$. This means $\mathrm{d}u = -\sin(t)$ and $\mathrm{d}v = -e^{-t}$.

$$\begin{aligned} \int e^{-t}\cos(t)\,\mathrm{d}t &= uv - \int v\cdot \mathrm{d}u \\ &= -\cos(t)e^{-t} - \int e^{-t}\cdot\sin(t)\,\mathrm{d}t \\ &= -\cos(t)e^{-t} - I \end{aligned}$$

Combining these results:

$$\begin{aligned} I &= -\sin(t)e^{-t} - \cos(t)e^{-t} - I \\ \Rightarrow 2I &= -\sin(t)e^{-t} - \cos(t)e^{-t} \\ \Rightarrow I &= \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2} \end{aligned}$$

Finally, we have:

$$I\Big|_0^{\infty} = \frac{0-0}{2} - \frac{0-1}{2} = \frac{1}{2}.$$

(b) We may set up the integral over the region R as follows:

$$\int_0^1 \int_0^x 2x + y \, \mathrm{d}y \, \mathrm{d}x.$$

Evaluating this integral gives

$$\begin{aligned}\int_0^1 \int_0^x 2x + y \, \mathrm{d}y \, \mathrm{d}x &= \int_0^1 2xy + \frac{y^2}{2}\Big|_0^x \mathrm{d}x \\ &= \int_0^1 \frac{5x^2}{2} \, \mathrm{d}x \\ &= \frac{5x^3}{6}\Big|_0^1 \\ &= \frac{5}{6}.\end{aligned}$$

2 More Logical Equivalences

Note 1

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a) $\forall x (P(x) \implies Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \implies \forall x\, Q(x)$

(b) $\neg(\exists x (P(x) \lor Q(x))) \stackrel{?}{\equiv} \forall x (\neg P(x) \land \neg Q(x))$

(c) $\forall x ((P(x) \implies Q(x)) \land Q(x)) \stackrel{?}{\equiv} \forall x\, P(x)$

Solution:

(a) Not Equivalent.

Justification: We can rewrite the left side as

$$\forall x (P(x) \implies Q(x)) \equiv \forall x (\neg P(x) \lor Q(x)).$$

We can also rewrite the right side as

$$\forall x\, P(x) \implies \forall x\, Q(x) \equiv \neg(\forall x P(x)) \lor \forall x\, Q(x) \equiv \exists x (\neg P(x)) \lor \forall x\, Q(x).$$

Then consider if $P(x) = \neg Q(x)$. Let $P(x)$ be true everywhere, thus $Q(x)$ must be false everywhere. Then, it is not the case that $\neg P(x)$ would be true anywhere, $\forall x\, Q(x)$ is false as it is false everywhere.

(b) Equivalent.

Justification:

$$\begin{aligned}\neg(\exists x(P(x)\vee Q(x))) &\equiv \exists x(\neg(P(x)\vee Q(x)) \\ &\equiv \forall x(\neg P(x))\wedge\forall x(\neg Q(x)) && \text{[De Morgan's Law]} \\ &\equiv \forall x(\neg P(x)\wedge\neg Q(x)).\end{aligned}$$

Intuitively, the LHS says that there is no x such that $P(x)\vee Q(x)$, so $\neg\exists x\, P(x)\vee\neg\exists x\, Q(x)$, so there is no x such that $P(x)$ is true nor an x such that $Q(x)$ are true; so $P(x)$ and $Q(x)$ are false everywhere. Then $\neg P(x)$ and $\neg Q(x)$ are true everywhere, and so $\neg P(x)\wedge\neg Q(x)$ is true everywhere which is the RHS.

(c) Not Equivalent.

Justification: Consider the case where $Q(x)$ is always true; for example, $Q(x) = x\in\mathbb{R}$, where the universe is the real numbers. Then regardless of the truth value of $P(x)$, $P(x) \implies Q(x)$ would be true, and thus the LHS would always be true.

Then we can choose some $P(x)$ which is not always true - e.g. $P(x) = x > 0$, so the RHS would be false, and the two sides are not equivalent.

3 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n\in\mathbb{N})$ if n is odd then n^2+4n is odd.

(b) $(\forall a,b\in\mathbb{R})$ if $a+b\leq 15$ then $a\leq 11$ or $b\leq 4$.

(c) $(\forall r\in\mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n\in\mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

(e) The product of a non-zero rational number and an irrational number is irrational.

(f) If $A\subseteq B$, then $\mathscr{P}(A)\subseteq\mathscr{P}(B)$. (Recall that $A'\in\mathscr{P}(A)$ if and only if $A'\subseteq A$.)

Solution:

(a) **Answer**: True.

Proof. We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. This means that we have

$$\begin{aligned}n^2+4n &= (2k+1)^2+4(2k+1) \\ &= 4k^2+12k+5 \\ &= 2(2k^2+6k+2)+1\end{aligned}$$

Since $2k^2+6k+2$ is a natural number, by the definition of odd numbers, n^2+4n is odd.

Alternatively, we could also factor the expression to get $n(n+4)$. Since n is odd, $n+4$ is also odd. The product of 2 odd numbers is also an odd number. Hence n^2+4n is odd. $\square$

(b) **Answer**: True.

Proof. We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg(a \leq 11 \vee b \leq 4)$). Since $a > 11$ and $b > 4$, $a+b > 15$ (note that $a+b > 15$ is equivalent to $\neg\big(a+b \leq 15\big)$). Thus, if $a+b \leq 15$, then $a \leq 11$ or $b \leq 4$. $\square$

(c) **Answer**: True.

Proof. We will use a proof by contraposition. Assume that r is rational. Since r is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$. Then r^2 can be written as $\frac{a^2}{b^2}$. By the definition of rational numbers, r^2 is a rational number, since both a^2 and b^2 are integers, with $b \neq 0$. By contraposition, if r^2 is irrational, then r is irrational. $\square$

(d) **Answer**: False.

Proof. We will show a counterexample. Let $n = 7$. Here, $5 \cdot 7^3 = 1715$, but $7! = 5040$. Since $5n^3 < n!$, the claim is false.

A counterexample that is easier to see without much calculation is for a much larger number like $n = 100$; here, $100!$ is clearly more than $5 \cdot 100^3 = 100 \cdot 50 \cdot 25 \cdot 5 \cdot 4 \cdot 2$, since the latter product contains only a subset of the terms in $100!$. $\square$

(e) **Answer**: True.

Proof. We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where p, q, r, and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational. $\square$

(f) **Answer**: True.

Proof. Suppose $A' \in \mathscr{P}(A)$; this means that $A' \subseteq A$ (by the definition of the power set).

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $(\forall x \in A')(x \in B)$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $(\forall A' \in \mathscr{P}(A))(A' \subseteq B)$. So, $(\forall A' \in \mathscr{P}(A))(A' \in \mathscr{P}(B))$ Thus, we conclude $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ as desired. $\square$

4 Twin Primes

Note 2

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Solution:

(a) First we note that any integer can be written in one of the forms $3k$, $3k+1$, or $3k+2$. (Note that $3k+2$ is equal to $3(k+1)-1$. Since k is arbitary, we can treat these as equivalent forms).

We can now prove the contrapositive: that any integer $m > 3$ of the form $3k$ must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5?

For any prime $m > 5$, we can check if $m+2$ and $m-2$ are both prime. Note that if $m > 5$, then $m+2 > 3$ and $m-2 > 3$ so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1: m is of the form $3k+1$. Then $m+2 = 3k+3$, which is divisible by 3. So $m+2$ is not prime.

Case 2: m is of the form $3k-1$. Then $m-2 = 3k-3$, which is divisible by 3. So $m-2$ is not prime.

So in either case, at least one of $m+2$ and $m-2$ is not prime.

5 Airport

Note 3

Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution: We proceed by induction on n. For $n = 1$, let the 3 airports be A, B, C and without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A.

Now suppose the statement holds for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports (say X and Y) must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k+1$ airports.

From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z. When we add back the two airports that we removed, there are two scenarios:

- Some of the flights get remapped to X or Y.

- None of the flights get remapped.

In either scenario, we conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$.

6 Grid Induction

Note 3

Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes).

Every second he does one of the following (if possible):

(i) Walk one step down, to $(i, j-1)$.

(ii) Walk one step left, to $(i-1, j)$.

For example, if he is at $(5,0)$, his only option is to walk left to $(4,0)$; if Pacman is instead at $(3,2)$, he could walk either to $(2,2)$ or $(3,1)$.

Prove by induction that no matter how he walks, he will always reach $(0,0)$ in finite time.

(*Hint*: Try starting Pacman at a few small points like $(2,1)$ and looking all the different paths he could take to reach $(0,0)$. Do you notice a pattern in the number of steps he takes? Try to use this to strengthen the inductive hypothesis.)

Solution: On first glance, this problem seems quite tricky, since we'd want to induct on *two* variables (i and j) rather than just one variable (as we've seen most commonly). However, following the hint, if we try out some smaller cases, we can notice that it takes Pacman $i+j$ seconds to reach $(0,0)$ if he starts in position (i, j), regardless what path he takes. This would imply that he reaches $(0,0)$ in a finite amount of time, since $i+j$ is a finite number.

This means that the quantity $i+j$ is something we could instead focus on, rather than the coordinate (i, j). In particular, we can try to induct on $i+j$ (essentially inducting on the amount of time it takes for Pacman to reach $(0,0)$), rather than inducting on i and j separately.

Proof. **Base Case**: If $i+j=0$, we know that $i=j=0$, since i and j must be non-negative. Hence, we have that Pacman is already at position $(0,0)$ and so will take $0=i+j$ steps to get there.

Inductive Hypothesis: Suppose that if Pacman starts at position (i, j) such that $i+j=n$, he will reach $(0,0)$ in finite time regardless of his path.

Inductive Step: Now suppose Pacman starts at position (i, j) such that $i+j=n+1$. If Pacman's first move is to position $(i-1, j)$, the sum of his x and y positions will be $i-1+j=(i+j)-1=n$. Thus, our inductive hypothesis tells us that it will take him a finite amount of time to get to $(0,0)$ no matter what path he takes. If Pacman's first move isn't to $(i-1, j)$, then it must be to $(i, j-1)$. Again in this case, the inductive hypothesis will tell us that Pacman will use a finite amount of time to get to $(0,0)$ no matter what path he takes. Thus, in either case, we have that Pacman will take a finite amount of time (one second for the first move and some additional finite time for the remainder) in order to reach $(0,0)$, proving the claim for $n+1$. $\square$

Note that once we had observed that it seems to take exactly $i+j$ seconds for Pacman to reach $(0,0)$ from (i,j), we could have tried to prove this stronger claim. This is equivalent to the above proof, with the only difference being the more specific length of time used in the inductive hypothesis; all other steps are identical.

One can also prove this statement without this trick inducting on $i+j$. The proof isn't quite as elegant, but is included here anyways for reference.

We first prove by induction on i that if Pacman starts from position $(i,0)$, he will reach $(0,0)$ in finite time.

Proof. **Base Case**: If $i=0$, Pacman starts at position $(0,0)$, so he doesn't need any more steps. Thus, it takes Pacman 0 steps to reach the origin, where 0 is a finite number.

Inductive Hypothesis: Suppose that if $i=n$ (that is, if Pacman starts at position $(n,0)$), he will reach $(0,0)$ in finite time.

Inductive Step: Now say Pacman starts at position $(n+1,0)$. Since he is on the x-axis, he has only one move: he has to move to $(n,0)$. From the inductive hypothesis, we know he will only take finite time to get to $(0,0)$ once he's gotten to $(n,0)$, so he'll only take a finite amount of time plus one second to get there from $(n+1,0)$. A finite amount of time plus one second is still a finite amount of time, so we've proved the claim for $i=n+1$. $\square$

We can now use this statement as the base case to prove our original claim by induction on j.

Proof. **Base Case**: If $j=0$, Pacman starts at position $(i,0)$ for some $i \in \mathbb{N}$. We proved above that Pacman must reach $(0,0)$ in finite time starting from here.

Inductive Hypothesis: Suppose that if Pacman starts in position (i,n), he'll reach $(0,0)$ in finite time no matter what i is.

Inductive Step: We now consider what happens if Pacman starts from position $(i,n+1)$, where i can be any natural number. If Pacman starts by moving down, we can immediately apply the inductive hypothesis, since Pacman will be in position (i,n). However, if Pacman moves to the left, he'll be in position $(i-1,n+1)$, so we can't yet apply the inductive hypothesis. But note that Pacman can't keep moving left forever: after i such moves, he'll hit the wall on the y-axis and be forced to move down. Thus, Pacman must make a vertical move after only finitely many horizontal moves–and once he makes that vertical move, he'll be in position (k,n) for some $0 \le k \le i$, so the inductive hypothesis tells us that it will only take him a finite amount of time to reach $(0,0)$ from there. This means that Pacman can only take a finite amount of time moving to the left, one second making his first move down, then a finite amount of additional time after his first vertical move.

Since a finite number plus one plus another finite number is still finite, this gives us our desired claim: Pacman must reach $(0,0)$ in finite time if he starts from position $(i, n+1)$ for any $i \in \mathbb{N}$. $\square$