1 Solving a System of Equations

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

**Solution:**
Letting $a$, $b$, and $c$ be the dollar cost of an apple, beet, and carrot, respectively, we get the system of equations

\[
\begin{align*}
    a + b + c &= 16 \\
    2a + 3b &= 23 \\
    a + 2b + 3c &= 35.
\end{align*}
\]

There are many approaches to solving this system (Gaussian Elimination, substitution, etc.). Here we show a solution via substitution.

Subtracting the third equation from three times the first equation gives

\[
2a + b = 3(a + b + c) - (a + 2b + 3c) = 3 \cdot 16 - 35 = 13.
\]

Subtracting this equation from the second equation gives

\[
2b = (2a + 3b) - (2a + b) = 23 - 13 = 10,
\]
so $b = 5$. Backsolving gives $a = 4$ and $c = 7$.

2 Calculus Review

(a) Compute the following integral:

\[
\int_0^\infty \sin(t)e^{-t} \, dt.
\]

(b) Find the minimum value of the following function over the reals and determine where it occurs.

\[
f(x) = \int_0^x e^{-t^2} \, dt.
\]

Show your work.
(c) Compute the double integral
\[ \iint_R (2x + y) \, dA, \]
where \( R \) is the region bounded by the lines \( x = 1, \ y = 0, \) and \( y = x. \)

Solution:

(a) Let \( I = \int \sin(t) e^{-t} \, dt. \) Use integration by parts, with \( u = \sin(t) \) and \( dv = e^{-t}. \) This means \( du = \cos(t) \) and \( v = -e^{-t}. \)

\[
I = \int \sin(t) e^{-t} \, dt = uv - \int v \cdot du = -\sin(t)e^{-t} + \int e^{-t} \cos(t) \, dt
\]

Use integration by parts again on \( \int e^{-t} \cos(t) \, dt, \) with \( u = \cos(t) \) and \( dv = e^{-t}. \) This means \( du = -\sin(t) \) and \( v = -e^{-t}. \)

\[
\int e^{-t} \cos(t) \, dt = uv - \int v \cdot du = -\cos(t)e^{-t} - \int e^{-t} \sin(t) \, dt
\]

\[
= -\cos(t)e^{-t} - I
\]

Combining these results:

\[
I = -\sin(t)e^{-t} - \cos(t)e^{-t} - I
\]

\[\Rightarrow 2I = -\sin(t)e^{-t} - \cos(t)e^{-t} \]

\[\Rightarrow I = \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2}
\]

Finally, we have:

\[
I \bigg|_0^\infty = \frac{0 - 0}{2} - \frac{0 - 1}{2} = \frac{1}{2}.
\]

(b) Compute the derivative of the function, and set it equal to 0. Let \( y = x^2. \) By the Chain Rule and the Fundamental Theorem of Calculus,

\[
\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} = e^{-x^4} \cdot 2x
\]

\[= 2xe^{-x^4} = 0\]

Since \( e^{-x^4} \) is always positive, we get that the derivative is 0 only at \( x^* = 0, \) with corresponding value of the function \( f(x^*) = 0. \)
We double check that the found point is a minima by checking the second derivative:

\[
\frac{d^2f}{dx^2} = \frac{d}{dx} 2xe^{-x^4} = 2e^{-x^4} - 8x^4 e^{-x^4}.
\]

The second derivative evaluated at \(x^* = 0\) is \(2e^{-0^4} - 8 \cdot 0^4 e^{-0^4} = 2\), which is positive, so our found point is a minima, as desired.

Hence, the minimum value of \(f(x)\) will be

\[
f(x^*) = f(0) = \int_0^0 e^{-t^2} dt = 0.
\]

(c) We may set up the integral over the region \(R\) as follows:

\[
\int_0^1 \int_0^x 2x + y dy dx.
\]

Evaluating this integral gives

\[
\int_0^1 \int_0^x 2x + y dy dx = \int_0^1 2xy + \frac{y^2}{2} \bigg|_0^x dx
\]

\[
= \int_0^1 5x^2 \bigg|_0^1 dx
\]

\[
= \frac{5x^3}{6} \bigg|_0^1
\]

\[
= \frac{5}{6}.
\]

3 Implication

Which of the following assertions are true no matter what proposition \(Q\) represents? For any false assertion, state a counterexample (i.e. come up with a statement \(Q(x,y)\) that would make the implication false). For any true assertion, give a brief explanation for why it is true.

(a) \(\exists x \exists y Q(x,y) \implies \exists y \exists x Q(x,y)\).

(b) \(\forall x \exists y Q(x,y) \implies \exists y \forall x Q(x,y)\).

(c) \(\exists x \forall y Q(x,y) \implies \forall y \exists x Q(x,y)\).

(d) \(\exists x \exists y Q(x,y) \implies \forall y \exists x Q(x,y)\).

Solution:

(a) True. There exists can be switched if they are adjacent; \(\exists x, \exists y\) and \(\exists y, \exists x\) means there exists \(x\) and \(y\) in our universe.
(b) False. Let $Q(x, y)$ be $x < y$, and the universe for $x$ and $y$ be the integers. Or let $Q(x, y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) True. The first statement says that there is an $x$, say $x'$ where for every $y$, $Q(x, y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every $y$. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

(d) False. Suppose $Q$ is the statement "$y$ is 5, and $x$ is any integer". The antecedent is true when $y = 5$, but for $y \neq 5$, there is no $x$ that will make it true.

4 Logical Equivalence?

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x \left( P(x) \land Q(x) \right) \equiv \forall x P(x) \land \forall x Q(x)$

(b) $\forall x \left( P(x) \lor Q(x) \right) \equiv \forall x P(x) \lor \forall x Q(x)$

(c) $\exists x \left( P(x) \lor Q(x) \right) \equiv \exists x P(x) \lor \exists x Q(x)$

(d) $\exists x \left( P(x) \land Q(x) \right) \equiv \exists x P(x) \land \exists x Q(x)$

Solution:

(a) Correct.

Assume that the left hand side is true. Then we know for an arbitrary $x$, $P(x) \land Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any $x$, $P(x)$ and for any $y$, $Q(y)$ holds, then for an arbitrary $x$ both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) Incorrect.

Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that $x$ can take on) is $\{1, 2\}$ and that $P$ and $Q$ are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false, $P(2)$ to be false and $Q(2)$ to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions $P$ and $Q$ for which these two expressions have different values, so they must be different.

(c) Correct

Assuming that the left hand side is true, we know there exists some $x$ such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exist an $x$ for which $P(x) \lor Q(x)$ is true, which means there is no $x$ for which $P(x)$ or $Q(x)$ is true. Therefore the right hand side is false.
(d) **Incorrect.**

Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that \(x\) can take on) is the natural numbers \(\mathbb{N}\), and that \(P\) and \(Q\) are truth functions defined on this universe. If we set \(P(1)\) to be true and \(P(x)\) to be false for all other \(x\), and \(Q(2)\) to be true and \(Q(x)\) to be false for all other \(x\), then the right hand side would be true. However, there would be no value of \(x\) at which both \(P(x)\) and \(Q(x)\) would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions \(P\) and \(Q\) for which these two expressions have different values, so they must be different.

5. **Preserving Set Operations**

For a function \(f\), define the image of a set \(X\) to be the set \(f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}\). Define the inverse image or preimage of a set \(Y\) to be the set \(f^{-1}(Y) = \{x \mid f(x) \in Y\}\). Prove the following statements, in which \(A\) and \(B\) are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets \(X\) and \(Y\), \(X = Y\) if and only if \(X \subseteq Y\) and \(Y \subseteq X\). To prove that \(X \subseteq Y\), it is sufficient to show that \((\forall x) ((x \in X) \implies (x \in Y))\).

(a) \(f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)\).

(b) \(f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)\).

(c) \(f(A \cap B) \subseteq f(A) \cap f(B)\), and give an example where equality does not hold.

(d) \(f(A \setminus B) \supseteq f(A) \setminus f(B)\), and give an example where equality does not hold.

**Solution:**

In order to prove equality \(A = B\), we need to prove that \(A\) is a subset of \(B\), \(A \subseteq B\) and that \(B\) is a subset of \(A\), \(B \subseteq A\). To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose \(x\) is such that \(f(x) \in A \cap B\). Then \(f(x)\) lies in both \(A\) and \(B\), so \(x\) lies in both \(f^{-1}(A)\) and \(f^{-1}(B)\), so \(x \in f^{-1}(A) \cap f^{-1}(B)\). So \(f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)\).

Now, suppose that \(x \in f^{-1}(A) \cap f^{-1}(B)\). Then, \(x\) is in both \(f^{-1}(A)\) and \(f^{-1}(B)\), so \(f(x) \in A\) and \(f(x) \in B\), so \(f(x) \in A \cap B\), so \(x \in f^{-1}(A \cap B)\). So \(f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)\).

(b) Suppose \(x\) is such that \(f(x) \in A \setminus B\). Then, \(f(x) \in A\) and \(f(x) \notin B\), which means that \(x \in f^{-1}(A)\) and \(x \notin f^{-1}(B)\), which means that \(x \in f^{-1}(A) \setminus f^{-1}(B)\). So \(f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)\).

Now, suppose that \(x \in f^{-1}(A) \setminus f^{-1}(B)\). Then, \(x \in f^{-1}(A)\) and \(x \notin f^{-1}(B)\), so \(f(x) \in A\) and \(f(x) \notin B\), so \(f(x) \in A \setminus B\), so \(x \in f^{-1}(A \setminus B)\). So \(f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)\).
(c) Suppose \( x \in A \cap B \). Then, \( x \) lies in both \( A \) and \( B \), so \( f(x) \) lies in both \( f(A) \) and \( f(B) \), so \( f(x) \in f(A) \cap f(B) \). Hence, \( f(A \cap B) \subseteq f(A) \cap f(B) \).

Consider when there are elements \( a \in A \) and \( b \in B \) with \( f(a) = f(b) \), but \( A \) and \( B \) are disjoint. Here, \( f(a) = f(b) \in f(A) \cap f(B) \), but \( f(A \cap B) \) is empty (since \( A \cap B \) is empty).

(d) Suppose \( y \in f(A) \setminus f(B) \). Since \( y \) is not in \( f(B) \), there are no elements in \( B \) which map to \( y \). Let \( x \) be any element of \( A \) that maps to \( y \); by the previous sentence, \( x \) cannot lie in \( B \). Hence, \( x \in A \setminus B \), so \( y \in f(A \setminus B) \). Hence, \( f(A) \setminus f(B) \subseteq f(A \setminus B) \).

Consider when \( B = \{0\} \) and \( A = \{0, 1\} \), with \( f(0) = f(1) = 0 \). One has \( A \setminus B = \{1\} \), so \( f(A \setminus B) = \{0\} \). However, \( f(A) = f(B) = \{0\} \), so \( f(A) \setminus f(B) = \emptyset \).

6 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) \( \forall n \in \mathbb{N} \) if \( n \) is odd then \( n^2 + 4n \) is odd.

(b) \( \forall a, b \in \mathbb{R} \) if \( a + b \leq 15 \) then \( a \leq 11 \) or \( b \leq 4 \).

(c) \( \forall r \in \mathbb{R} \) if \( r^2 \) is irrational, then \( r \) is irrational.

(d) \( \forall n \in \mathbb{Z}^+ \) \( 5n^3 > n! \). (Note: \( \mathbb{Z}^+ \) is the set of positive integers)

Solution:

(a) Answer: True.

Proof: We will use a direct proof. Assume \( n \) is odd. By the definition of odd numbers, \( n = 2k + 1 \) for some natural number \( k \). Substituting into the expression \( n^2 + 4n \), we get \( (2k + 1)^2 + 4 \cdot (2k + 1) \). Simplifying the expression yields \( 4k^2 + 12k + 5 \). This can be rewritten as \( 2 \cdot (2k^2 + 6k + 2) + 1 \). Since \( 2k^2 + 6k + 2 \) is a natural number, by the definition of odd numbers, \( n^2 + 4n \) is odd.

Alternatively, we could also factor the expression to get \( n(n + 4) \). Since \( n \) is odd, \( n + 4 \) is also odd. The product of 2 odd numbers is also an odd number. Hence \( n^2 + 4n \) is odd.

(b) Answer: True.

Proof: We will use a proof by contraposition. Suppose that \( a > 11 \) and \( b > 4 \) (note that this is equivalent to \( \neg(a \leq 11 \lor b \leq 4) \)). Since \( a > 11 \) and \( b > 4 \), \( a + b > 15 \) (note that \( a + b > 15 \) is equivalent to \( \neg(a + b \leq 15) \)). Thus, if \( a + b \leq 15 \), then \( a \leq 11 \) or \( b \leq 4 \).

(c) Answer: True.

Proof: We will use a proof by contraposition. Assume that \( r \) is rational. Since \( r \) is rational, it can be written in the form \( \frac{a}{b} \) where \( a \) and \( b \) are integers with \( b \neq 0 \). Then \( r^2 \) can be written as \( \frac{a^2}{b^2} \). By the definition of rational numbers, \( r^2 \) is a rational number, since both \( a^2 \) and \( b^2 \) are integers, with \( b \neq 0 \). By contraposition, if \( r^2 \) is irrational, then \( r \) is irrational.
(d) **Answer:** False.

**Proof:** We will show a counterexample. Let \( n = 7 \). \( 5^3 = 1715 \). \( 7! = 5040 \). Since \( 5n^3 < n! \), the claim is false.

### 7 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

**Solution:** We prove the statement by contradiction. Suppose that \( ab = c \), where \( a \neq 0 \) is rational, \( b \) is irrational, and \( c \) is rational. Since \( a \) and \( b \) are not zero (because 0 is rational), \( c \) is also non-zero. Thus, we can express \( a = \frac{p}{q} \) and \( c = \frac{r}{s} \), where \( p, q, r, \) and \( s \) are nonzero integers. Then

\[
\frac{b}{a} = \frac{rq}{ps},
\]

which is the ratio of two nonzero integers, giving that \( b \) is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational.

### 8 Twin Primes

(a) Let \( p > 3 \) be a prime. Prove that \( p \) is of the form \( 3k + 1 \) or \( 3k − 1 \) for some integer \( k \).

(b) **Twin primes** are pairs of prime numbers \( p \) and \( q \) that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

**Solution:**

(a) First we note that any integer can be written in one of the forms \( 3k, 3k + 1, \) or \( 3k + 2 \). (Note that \( 3k + 2 \) is equal to \( 3(k + 1) − 1 \). Since \( k \) is arbitrary, we can treat these as equivalent forms).

We can now prove the contrapositive: that any integer \( m > 3 \) of the form \( 3k \) must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5?

For any prime \( m > 5 \), we can check if \( m + 2 \) and \( m − 2 \) are both prime. Note that if \( m > 5 \), then \( m + 2 > 3 \) and \( m − 2 > 3 \) so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1: \( m \) is of the form \( 3k + 1 \). Then \( m + 2 = 3k + 3 \), which is divisible by 3. So \( m + 2 \) is not prime.

Case 2: \( m \) is of the form \( 3k − 1 \). Then \( m − 2 = 3k − 3 \), which is divisible by 3. So \( m − 2 \) is not prime.

So in either case, at least one of \( m + 2 \) and \( m − 2 \) is not prime.