

1 Logical Equivalence?

Note 1 Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \forall x P(x) \wedge \forall x Q(x)$

(b) $\forall x (P(x) \vee Q(x)) \stackrel{?}{\equiv} \forall x P(x) \vee \forall x Q(x)$

(c) $\exists x (P(x) \vee Q(x)) \stackrel{?}{\equiv} \exists x P(x) \vee \exists x Q(x)$

(d) $\exists x (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \exists x P(x) \wedge \exists x Q(x)$

Solution:

(a) **Correct.**

Assume that the left hand side is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) **Incorrect.**

Note that there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that x can take on) is $\{1, 2\}$ and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false, $P(2)$ to be false and $Q(2)$ to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

Another more concrete example is if $P(x) = x < 0$ and $Q(x) = x \geq 0$, where the universe is the real numbers. For any $x \in \mathbb{R}$, exactly one of $P(x)$ or $Q(x)$ is true, but it is not the case that $P(x)$ holds for every x , and it is also not the case that $Q(x)$ holds for every x . Since the LHS and RHS have different values, the two sides are not equivalent.

(c) **Correct**

Assuming that the left hand side is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exist an x for which $P(x) \vee$

$Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the right hand side is false.

(d) **Incorrect.**

Note, there are many possible counterexamples not described here.

Suppose that the universe (i.e. the values that x can take on) is the natural numbers \mathbb{N} , and that P and Q are truth functions defined on this universe. Here, suppose we set $P(1)$ to be true and $P(x)$ to be false for all other x , and $Q(2)$ to be true and $Q(x)$ to be false for all other x . (In other words, $P(x) = (x = 1)$ and $Q(x) = (x = 2)$.)

With these definitions, the right hand side would be true, since there exists some value of x that makes $P(x)$ true (namely, $x = 1$), and there exists some value of x that makes $Q(x)$ true (namely, $x = 2$). However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

2 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

- (a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.
- (b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.
- (c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.
- (d) $(\forall n \in \mathbb{Z}^+) 5n^3 > n!$. (Note: \mathbb{Z}^+ is the set of positive integers)
- (e) The product of a non-zero rational number and an irrational number is irrational.

Solution:

(a) **Answer:** True.

Proof. We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k . This means that we have

$$\begin{aligned} n^2 + 4n &= (2k + 1)^2 + 4(2k + 1) \\ &= 4k^2 + 12k + 5 \\ &= 2(2k^2 + 6k + 2) + 1 \end{aligned}$$

Since $2k^2 + 6k + 2$ is a natural number, by the definition of odd numbers, $n^2 + 4n$ is odd.

Alternatively, we could also factor the expression to get $n(n + 4)$. Since n is odd, $n + 4$ is also odd. The product of 2 odd numbers is also an odd number. Hence $n^2 + 4n$ is odd. \square

(b) **Answer:** True.

Proof. We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg(a \leq 11 \vee b \leq 4)$). Since $a > 11$ and $b > 4$, $a + b > 15$ (note that $a + b > 15$ is equivalent to $\neg(a + b \leq 15)$). Thus, if $a + b \leq 15$, then $a \leq 11$ or $b \leq 4$. \square

(c) **Answer:** True.

Proof. We will use a proof by contraposition. Assume that r is rational. Since r is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$. Then r^2 can be written as $\frac{a^2}{b^2}$. By the definition of rational numbers, r^2 is a rational number, since both a^2 and b^2 are integers, with $b \neq 0$. By contraposition, if r^2 is irrational, then r is irrational. \square

(d) **Answer:** False.

Proof. We will show a counterexample. Let $n = 7$. Here, $5 \cdot 7^3 = 1715$, but $7! = 5040$. Since $5n^3 < n!$, the claim is false.

A counterexample that is easier to see without much calculation is for a much larger number like $n = 100$; here, $100!$ is clearly more than $5 \cdot 100^3 = 100 \cdot 50 \cdot 25 \cdot 5 \cdot 4 \cdot 2$, since the latter product contains only a subset of the terms in $100!$. \square

(e) **Answer:** True.

Proof. We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where p, q, r , and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational. \square

3 Twin Primes

Note 2

- (a) Let $p > 3$ be a prime. Prove that p is of the form $3k + 1$ or $3k - 1$ for some integer k .
- (b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Solution:

(a) First we note that any integer can be written in one of the forms $3k$, $3k + 1$, or $3k + 2$. (Note that $3k + 2$ is equal to $3(k + 1) - 1$. Since k is arbitrary, we can treat these as equivalent forms). We can now prove the contrapositive: that any integer $m > 3$ of the form $3k$ must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5 ?

For any prime $m > 5$, we can check if $m + 2$ and $m - 2$ are both prime. Note that if $m > 5$, then $m + 2 > 3$ and $m - 2 > 3$ so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1: m is of the form $3k + 1$. Then $m + 2 = 3k + 3$, which is divisible by 3. So $m + 2$ is not prime.

Case 2: m is of the form $3k - 1$. Then $m - 2 = 3k - 3$, which is divisible by 3. So $m - 2$ is not prime.

So in either case, at least one of $m + 2$ and $m - 2$ is not prime.

4 Airport

Note 3 Suppose that there are $2n + 1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution: We proceed by induction on n . For $n = 1$, let the 3 airports be A, B, C and without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A .

Now suppose the statement holds for $n = k$, i.e. when there are $2k + 1$ airports. For $n = k + 1$, i.e. when there are $2k + 3$ airports, the airplanes departing from the closest two airports (say X and Y) must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k + 1$ airports.

From the inductive hypothesis, we know that among the $2k + 1$ airports remaining, there is an airport with no incoming flights which we call airport Z . When we add back the two airports that we removed, there are two scenarios:

- Some of the flights get remapped to X or Y .
- None of the flights get remapped.

In either scenario, we conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k + 1$. By induction, the claim holds for all $n \geq 1$.

5 A Coin Game

Note 3 Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n .

Base Case: If $n = 1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero—and indeed, $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$.

Inductive Step: Suppose that if you start with i coins (for i between 1 and n inclusive), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ. Now suppose you start with $n + 1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1 + s_2 = n + 1$ and $s_1, s_2 \geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get $s_1 s_2$ points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$, regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned} s_1 s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1 s_2 + s_2(s_2-1)}{2} \\ &= \frac{(s_1(s_1-1) + s_1 s_2) + (s_2(s_2-1) + s_1 s_2)}{2} \\ &= \frac{s_1(s_1 + s_2 - 1) + s_2(s_1 + s_2 - 1)}{2} \\ &= \frac{(s_1 + s_2)(s_1 + s_2 - 1)}{2} \end{aligned}$$

Since $s_1 + s_2 = n + 1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

6 Grid Induction

Note 3 Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes).

Every second he does one of the following (if possible):

- (i) Walk one step down, to $(i, j - 1)$.
- (ii) Walk one step left, to $(i - 1, j)$.

For example, if he is at $(5, 0)$, his only option is to walk left to $(4, 0)$; if Pacman is instead at $(3, 2)$, he could walk either to $(2, 2)$ or $(3, 1)$.

Prove by induction that no matter how he walks, he will always reach $(0, 0)$ in finite time.

(*Hint*: Try starting Pacman at a few small points like $(2, 1)$ and looking all the different paths he could take to reach $(0, 0)$. Do you notice a pattern in the number of steps he takes? Try to use this to strengthen the inductive hypothesis.)

Solution: On first glance, this problem seems quite tricky, since we'd want to induct on *two* variables (i and j) rather than just one variable (as we've seen most commonly). However, following the hint, if we try out some smaller cases, we can notice that it takes Pacman $i + j$ seconds to reach $(0, 0)$ if he starts in position (i, j) , regardless what path he takes. This would imply that he reaches $(0, 0)$ in a finite amount of time, since $i + j$ is a finite number.

This means that the quantity $i + j$ is something we could instead focus on, rather than the coordinate (i, j) . In particular, we can try to induct on $i + j$ (essentially inducting on the amount of time it takes for Pacman to reach $(0, 0)$), rather than inducting on i and j separately.

Proof. **Base Case:** If $i + j = 0$, we know that $i = j = 0$, since i and j must be non-negative. Hence, we have that Pacman is already at position $(0, 0)$ and so will take $0 = i + j$ steps to get there.

Inductive Hypothesis: Suppose that if Pacman starts at position (i, j) such that $i + j = n$, he will reach $(0, 0)$ in finite time regardless of his path.

Inductive Step: Now suppose Pacman starts at position (i, j) such that $i + j = n + 1$. If Pacman's first move is to position $(i - 1, j)$, the sum of his x and y positions will be $i - 1 + j = (i + j) - 1 = n$. Thus, our inductive hypothesis tells us that it will take him a finite amount of time to get to $(0, 0)$ no matter what path he takes. If Pacman's first move isn't to $(i - 1, j)$, then it must be to $(i, j - 1)$. Again in this case, the inductive hypothesis will tell us that Pacman will use a finite amount of time to get to $(0, 0)$ no matter what path he takes. Thus, in either case, we have that Pacman will take a finite amount of time (one second for the first move and some additional finite time for the remainder) in order to reach $(0, 0)$, proving the claim for $n + 1$. \square

Note that once we had observed that it seems to take exactly $i + j$ seconds for Pacman to reach $(0, 0)$ from (i, j) , we could have tried to prove this stronger claim. This is equivalent to the above proof, with the only difference being the more specific length of time used in the inductive hypothesis; all other steps are identical.

One can also prove this statement without this trick inducting on $i + j$. The proof isn't quite as elegant, but is included here anyways for reference.

We first prove by induction on i that if Pacman starts from position $(i, 0)$, he will reach $(0, 0)$ in finite time.

Proof. Base Case: If $i = 0$, Pacman starts at position $(0, 0)$, so he doesn't need any more steps. Thus, it takes Pacman 0 steps to reach the origin, where 0 is a finite number.

Inductive Hypothesis: Suppose that if $i = n$ (that is, if Pacman starts at position $(n, 0)$), he will reach $(0, 0)$ in finite time.

Inductive Step: Now say Pacman starts at position $(n + 1, 0)$. Since he is on the x -axis, he has only one move: he has to move to $(n, 0)$. From the inductive hypothesis, we know he will only take finite time to get to $(0, 0)$ once he's gotten to $(n, 0)$, so he'll only take a finite amount of time plus one second to get there from $(n + 1, 0)$. A finite amount of time plus one second is still a finite amount of time, so we've proved the claim for $i = n + 1$. \square

We can now use this statement as the base case to prove our original claim by induction on j .

Proof. Base Case: If $j = 0$, Pacman starts at position $(i, 0)$ for some $i \in \mathbb{N}$. We proved above that Pacman must reach $(0, 0)$ in finite time starting from here.

Inductive Hypothesis: Suppose that if Pacman starts in position (i, n) , he'll reach $(0, 0)$ in finite time no matter what i is.

Inductive Step: We now consider what happens if Pacman starts from position $(i, n + 1)$, where i can be any natural number. If Pacman starts by moving down, we can immediately apply the inductive hypothesis, since Pacman will be in position (i, n) . However, if Pacman moves to the left, he'll be in position $(i - 1, n + 1)$, so we can't yet apply the inductive hypothesis. But note that Pacman can't keep moving left forever: after i such moves, he'll hit the wall on the y -axis and be forced to move down. Thus, Pacman must make a vertical move after only finitely many horizontal moves—and once he makes that vertical move, he'll be in position (k, n) for some $0 \leq k \leq i$, so the inductive hypothesis tells us that it will only take him a finite amount of time to reach $(0, 0)$ from there. This means that Pacman can only take a finite amount of time moving to the left, one second making his first move down, then a finite amount of additional time after his first vertical move. Since a finite number plus one plus another finite number is still finite, this gives us our desired claim: Pacman must reach $(0, 0)$ in finite time if he starts from position $(i, n + 1)$ for any $i \in \mathbb{N}$. \square

7 (Optional) Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:

$$\int_0^{\infty} \sin(t)e^{-t} dt.$$

(b) Compute the values of $x \in (-2, 2)$ that correspond to local maxima and minima of the function

$$f(x) = \int_0^{x^2} t \cos(\sqrt{t}) dt.$$

Classify which x correspond to local maxima and which to local minima.

(c) Compute the double integral

$$\iint_R 2x + y dA,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

Solution:

(a) Let $I = \int \sin(t)e^{-t} dt$.

Use integration by parts, with $u = \sin(t)$ and $dv = e^{-t}$.

This means $du = \cos(t)$ and $v = -e^{-t}$.

$$\begin{aligned} I &= \int \sin(t)e^{-t} dt = uv - \int v \cdot du \\ &= -\sin(t)e^{-t} + \int e^{-t} \cos(t) dt \end{aligned}$$

Use integration by parts again on $\int e^{-t} \cos(t) dt$, with $u = \cos(t)$ and $dv = e^{-t}$. This means $du = -\sin(t)$ and $dv = -e^{-t}$.

$$\begin{aligned} \int e^{-t} \cos(t) dt &= uv - \int v \cdot du \\ &= -\cos(t)e^{-t} - \int e^{-t} \cdot \sin(t) dt \\ &= -\cos(t)e^{-t} - I \end{aligned}$$

Combining these results:

$$\begin{aligned} I &= -\sin(t)e^{-t} - \cos(t)e^{-t} - I \\ \Rightarrow 2I &= -\sin(t)e^{-t} - \cos(t)e^{-t} \\ \Rightarrow I &= \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2} \end{aligned}$$

Finally, we have:

$$I \Big|_0^{\infty} = \frac{0-0}{2} - \frac{0-1}{2} = \frac{1}{2}.$$

- (b) Compute the derivative of the function, and set it equal to 0. Let $y = x^2$. By the Chain Rule and the Fundamental Theorem of Calculus,

$$\begin{aligned}\frac{df}{dx} &= \frac{df}{dy} \cdot \frac{dy}{dx} \\ &= y \cos(\sqrt{y}) \cdot 2x \\ &= 2x^3 \cos(|x|) \\ &= 2x^3 \cos(x) = 0\end{aligned}$$

We get that the derivative is 0 only when $x^* = 0$, or when $\cos(x^*) = 0$. On the interval $(-2, 2)$, this corresponds to critical points $-\pi/2, 0$, and $\pi/2$.

To classify which correspond to local maxima and which to local minima, we examine how the sign of the derivative changes.

Around $x = \pi/2$, the derivative is positive for $x < \pi/2$ and negative for $x > \pi/2$. The same holds for $x = -\pi/2$. Thus, $x = \pm\pi/2$ correspond to local maxima.

Around $x = 0$, the derivative is negative for $x < 0$ and positive for $x > 0$. Thus, $x = 0$ corresponds to a local minima.

- (c) We may set up the integral over the region R as follows:

$$\int_0^1 \int_0^x 2x + y \, dy \, dx.$$

Evaluating this integral gives

$$\begin{aligned}\int_0^1 \int_0^x 2x + y \, dy \, dx &= \int_0^1 2xy + \frac{y^2}{2} \Big|_0^x \, dx \\ &= \int_0^1 \frac{5x^2}{2} \, dx \\ &= \frac{5x^3}{6} \Big|_0^1 \\ &= \frac{5}{6}.\end{aligned}$$