

Due: Saturday, 2/7, 4:00 PM
Grace period until Saturday, 2/7, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Airport

Note 3 Suppose that there are $2n + 1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that, for all $n > 0$, there is an airport which has no airplanes destined for it. (Hint: In the inductive step, it is not sufficient to start at a $2n + 1$ airport example and add two airports to create a $2(n + 1) + 1$ airport example. Think of why this doesn't generate **all** $2(n + 1) + 1$ airport examples, and what you can do to fix this. Consider taking a look at **#2 on Discussion 2A!**)

2 Universal Preference

Note 4 Suppose that preferences in a stable matching instance are universal, i.e., all n jobs share the preferences $C_1 > C_2 > \dots > C_n$ and all candidates share the preferences $J_1 > J_2 > \dots > J_n$.

- What pairing do we get from running the propose-and-reject algorithm with jobs proposing? (Hint: Start with small examples and go through the algorithm. Do you see a pattern?) Prove that this happens for all n .
- What pairing do we get from running the propose-and-reject algorithm with candidates proposing? Explain.
- What does this tell us about the number of stable pairings? Justify your answer.

3 Pairing Up

Note 4 Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

(*Hint:* It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n . Additionally, $2^{n/2}$ is a very specific number; try to think about how your construction would build such a number as n increases.)

4 Short Tree Proofs

Note 5 Let $G = (V, E)$ be an undirected graph with $|V| \geq 1$.

- Prove that every connected component in an acyclic graph is a tree.
- Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.
- Prove that a graph with $|V|$ edges contains a cycle.

5 Proofs in Graphs

Note 5 (a) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof. Also, note that Euler's Theorem holds for both simple graphs and multigraphs (graphs where multiple edges between one pair of vertices is permitted).]

(b) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of the shortest paths from some fixed vertex to all the other vertices.]

6 Self-Grades

Make sure to review the self grades post on Edstem and submit your selfgrades for the previous homework assignment on Gradescope! This is just a reminder to do so, no need to submit anything for this question.