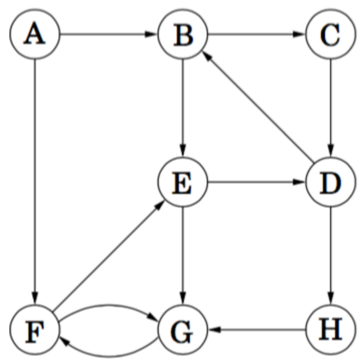
CS 70 Discrete Mathematics and Probability Theory
Fall 2021

# HW 3

## 1 Graph Basics

In the first few parts, you will be answering questions on the following graph $G$.




(a) What are the vertex and edge sets $V$ and $E$ for graph $G$?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex $B$ to $F$, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in $G$?

i. $(B,C),(C,D),(D,B)$

ii. $(F,G),(G,F)$

iii. $(A,B),(B,C),(C,D),(D,B)$

iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in $G$?

i. $(E,G)$

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(A,B),(B,C),(C,D),(H,G)$

v. $(E,G),(G,F),(F,G),(G,C)$

vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in $G$?

i. $(E,G)$

ii. $(E,G),(G,F)$

iii. $(F,G),(G,F)$

iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

v. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

**In the following three parts, let's consider a general undirected graph $G$ with $n$ vertices ($n \geq 3$).** If true, provide a short proof. If false, show a counterexample.

(g) True/False: If each vertex of $G$ has degree at most 1, then $G$ does not have a cycle.

(h) True/False: If each vertex of $G$ has degree at least 2, then $G$ has a cycle.

(i) True/False: If each vertex of $G$ has degree at most 2, then G is not connected.

**Solution:**

(a) A graph is specified as an ordered pair $G = (V,E)$, where $V$ is the vertex set and $E$ is the edge set.

$$V = \{A,B,C,D,E,F,G,H\},$$
$$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),(H,G)\}.$$

(b) $G$ has the highest in-degree (3). $A$ has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. $H$ and $C$ has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$(B,C),(C,D),(D,H),(H,G),(G,F)$
$(B,E),(E,D),(D,H),(H,G),(G,F)$
$(B,E),(E,G),(G,F)$
The first two have length 5, while the last one has length 3, so the last one is the shortest path.

(d) A cycle is a path that starts and ends at the same point. This means that (iii) is not a cycle, since it starts at $A$ but ends at $B$. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)$ should be distinct, so (iv) is not a cycle. The correct answers are (i) and (ii).

(e) A walk consists of any sequence of edges such that the endpoint of each edge is the same as the starting vertex of the next edge in the sequence. Example (iv) does not fit this definition–even though it uses only valid edges, the endpoint of the second to last edge in $D$, while the start point of the next edge is $H$. Example (v) also is not a walk, since it tries to walk from $G$ to $C$ as its last step, but there is no such edge. All the rest are walks.

(f) A tour is a walk that has the same start and end vertex; examples (iii) and (v) satisfy this definition. Note in part (d), we already said that (iii) was a cycle–and indeed, all cycles are also tours.

(g) True. In order for there to be a cycle in $G$ starting and ending at some vertex $v$, we would need at least two edges incident to $v$: one to leave $v$ at the start of the cycle, and one to return to $v$ at the end. If every vertex has degree at most 1, no vertex has two or more edges incident on it, so no vertex is capable of acting as the start and end point of a cycle.

(h) True. Consider starting a walk at some vertex $v_0$, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex $w$ for the second time. If this process terminates, the part of our walk from the first time we visited $w$ until the second time is a cycle. Thus, it remains only to argue this process always terminates.

Each time we take a step from some vertex $v$, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with $v$ (either we started at $v$, or we took an edge into $v$). Since $v$ has degree at least 2, there must be another edge leaving $v$ for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

## 2 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets $L$, $R$ such that each edge connects a vertex in $L$ to a vertex in $R$ (so there does not exist an edge that connects two vertices in $L$ or two vertices in $R$).

(a) Suppose that a graph $G$ is bipartite, with $L$ and $R$ being a bipartite partition of the vertices. Prove that $\sum_{v\in L}\deg(v) = \sum_{v\in R}\deg(v)$.

(b) Suppose that a graph $G$ is bipartite, with $L$ and $R$ being a bipartite partition of the vertices. Let $s$ and $t$ denote the average degree of vertices in $L$ and $R$ respectively. Prove that $s/t = |R|/|L|$.

(c) Prove that a graph is bipartite if and only if it can be 2-colored. (A graph can be 2-colored if every vertex can be assigned one of two colors such that no two adjacent vertices have the same color).

**Solution:**

(a) Since $G$ is bipartite, each edge connects one vertex in $L$ with a vertex in $R$. Since each edge contributes equally to $\sum_{v\in L}\deg(v)$ and $\sum_{v\in R}\deg(v)$, we see that these two values must be equal.

(b) By part (a), we know that $\sum_{v\in L}\deg(v) = \sum_{v\in R}\deg(v)$. Thus $|L|\cdot s = |R|\cdot t$. A little algebra gives us the desired result.

(c) Given a bipartite graph, color all of the vertices in $L$ one color, and all of the vertices in $R$ the other color. Conversely, given a 2-colored graph (call the colors red and blue), there are no edges between red vertices and red vertices, and there are no edges between blue vertices and blue vertices. Hence, take $L$ to be the set of red vertices and $R$ to be the set of blue vertices. We see that the graph is bipartite.

## 3 Touring Hypercube

In the lecture, you have seen that if $G$ is a hypercube of dimension $n$, then

- The vertices of $G$ are the binary strings of length $n$.
- $u$ and $v$ are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- $v_0$ and $v_k$ are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if $n$ is even. (*Hint: Euler's theorem*)

(b) Show that every hypercube has a Hamiltonian tour.

**Solution:**

(a) In the $n$-dimensional hypercube, every vertex has degree $n$. If $n$ is odd, then by Euler's Theorem there can be no Eulerian tour. On the other hand, the hypercube is connected: we can get from any one bit-string $x$ to any other $y$ by flipping the bits they differ in one at a time. Therefore, when $n$ is even, since every vertex has even degree and the graph is connected, there is an Eulerian tour.

(b) By induction on $n$. When $n = 1$, there are two vertices connected by an edge; we can form a Hamiltonian tour by walking from one to the other and then back.

Let $n \geq 1$ and suppose the $n$-dimensional hypercube has a Hamiltonian tour. Let $H$ be the $n+1$-dimensional hypercube, and let $H_b$ be the $n$-dimensional subcube consisting of those strings with initial bit $b$.

By the inductive hypothesis, there is some Hamiltonian tour $T$ on the $n$-dimensional hypercube. Now consider the following tour in $H$. Start at an arbitrary vertex $x_0$ in $H_0$, and follow the tour $T$ except for the very last step to vertex $y_0$ (so that the next step would bring us back to $x_0$). Next take the edge from $y_0$ to $y_1$ to enter cube $H_1$. Next, follow the tour $T$ in $H_1$ backwards from $y_1$, except the very last step, to arrive at $x_1$. Finally, take the step from $x_1$ to $x_0$ to complete the tour. By assumption, the tour $T$ visits each vertex in each subcube exactly once, so our complete tour visits each vertex in the whole cube exactly once.

To build some intuition, here are the first few cases:

- $n = 1$: 0, 1
- $n = 2$: 00, 01, 11, 10 [Take the $n = 1$ tour in the 0-subcube (vertices with a 0 in front), move to the 1-subcube (vertices with 1 in front), then take the tour backwards. We know 10 connects to 00 to complete the tour.]
- $n = 3$: 000, 001, 011, 010, 110, 111, 101, 100 [Take the $n = 2$ tour in the 0-subcube, move to the 1-subcube, then take the tour backwards. We know 100 connects to 000 to complete the tour.]

The sequence produced with this method is known as a Gray code.

# 4 Tournament

A *tournament* is defined to be a directed graph such that for every pair of distinct nodes $v$ and $w$, exactly one of $(v,w)$ and $(w,v)$ is an edge (representing which player beat the other in a round-robin tournament). Prove that every tournament has a Hamiltonian path. In other words, you can always arrange the players in a line so that each player beats the next player in the line.

**Solution:**

We provide two possible answers: one using simple induction, and the other using strong induction.

**Answer 1:** We will prove this with induction on the number of nodes/players.

*Base Case* $n = 1$*:* There is only one player, so the claim is trivially true.

*Inductive Hypothesis:* Suppose for some $n \geq 1$, we can find a Hamiltonian path in a tournament of $n$ players.

*Inductive Step:* Consider a tournament of $n+1$ players. Arbitrarily pick one player $p_{n+1}$ to "hold out." From our inductive hypothesis, we can arrange the remaining $n$ players in a line, say $p_1, p_2, \ldots, p_n$, such that $p_i$ beat $p_{i+1}$ for $1 \leq i \leq n-1$.

Let $p_a$ be the last player that beat $p_{n+1}$. If there is no such $p_a$ (i.e., $p_{n+1}$ beat everyone), then we can place $p_{n+1}$ before $p_1$, and we are done. Otherwise, reorder the players as follows:

$$p_1, p_2, \ldots, p_a, p_{n+1}, p_{a+1}, \ldots, p_n.$$

We know that $p_{n+1}$ must have beaten $p_{a+1}$ by definition (or else $p_{a+1}$ would be the last player that beat $p_{n+1}$). If it turns out that $a = n$, we simply place $p_{n+1}$ after $p_n$, and we still have a valid Hamiltonian path.

Therefore, for all $n \geq 1$, there exists a Hamiltonian path in a tournament of $n$ players. $\square$

**Answer 2:** We will prove this with strong induction on the number of nodes/players.

*Base Case* $n = 1$*:* There is only one player, so the claim is trivially true.

*Inductive Hypothesis:* Suppose for all $1 \le k < n$, we can find a Hamiltonian path in a tournament of $k$ players.

*Inductive Step:* Consider a tournament of $n$ players.

Arbitrarily pick one player $p$ to "hold out." Let $S$ be the set of players who beat $p$, and let $T$ be the set of players who $p$ beat. From our inductive hypothesis, we can find a Hamiltonian path in $S$ and in $T$. Finally, to obtain a Hamiltonian path on all $n$ players, we connect the last person in $S$ to $p$, and $p$ to the first person in $T$.

Therefore, for all $n \ge 1$, there exists a Hamiltonian path in a tournament of $n$ players. □

# 5 Planarity and Graph Complements

Let $G = (V,E)$ be an undirected graph. We define the complement of $G$ as $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{(i,j) | i,j \in V, i \ne j\} - E$; that is, $\overline{G}$ has the same set of vertices as $G$, but an edge $e$ exists is $\overline{G}$ if and only if it does not exist in $G$.

(a) Suppose $G$ has $v$ vertices and $e$ edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, $G$ being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph $G$ with at least 13 vertices, if $\overline{G}$ is non-planar, then $G$ is planar. Construct a counterexample to show that the converse does not hold.

*Hint: Recall that if a graph contains a copy of $K_5$, then it is non-planar. Can this fact be used to construct a counterexample?*

**Solution:**

(a) If $G$ has $v$ vertices, then there are a total of $\frac{v(v-1)}{2}$ edges that could possibly exist in the graph. Since $e$ of them appear in $G$, we know that the remaining $\frac{v(v-1)}{2} - e$ must appear in $\overline{G}$.

(b) Since $G$ is planar, we know that $e \le 3v - 6$. Plugging this in to the answer from the previous part, we have that $\overline{G}$ has at least $\frac{v(v-1)}{2} - (3v-6)$ edges. Since $v$ is at least 13, we have that $\frac{v(v-1)}{2} \ge \frac{v \cdot 12}{2} = 6v$, so $\overline{G}$ has at least $6v - 3v + 6 = 3v + 6$ edges. Since this is strictly more than the $3v - 6$ edges allowed in a planar graph, we have that $\overline{G}$ must not be planar.

(c) The converse is not necessarily true. As a counterexample, suppose that $G$ has exactly thirteen vertices, of which five are all connected to each other and the remaining eight have no edges incident to them. This means that $G$ is non-planar, since it contains a copy of $K_5$. However, $\overline{G}$ also contains a copy of $K_5$ (take any 5 of the 8 vertices that were isolated in $G$), so $\overline{G}$ is also non-planar. Thus, it is possible for both $G$ and $\overline{G}$ to be non-planar.

## 6 Build-Up Error?

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger build-up error.

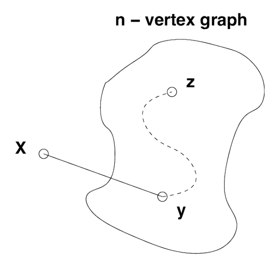
**False Claim:** If every vertex in an undirected graph has degree at least 1, then the graph is connected.

*Proof:* We use induction on the number of vertices $n \geq 1$.

*Base case:* There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

*Inductive hypothesis:* Assume the claim is true for some $n \geq 1$.

*Inductive step:* We prove the claim is also true for $n+1$. Consider an undirected graph on $n$ vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex $x$ to obtain a graph on $(n+1)$ vertices, as shown below.




All that remains is to check that there is a path from $x$ to every other vertex $z$. Since $x$ has degree at least 1, there is an edge from $x$ to some other vertex; call it $y$. Thus, we can obtain a path from $x$ to $z$ by adjoining the edge $\{x,y\}$ to the path from $y$ to $z$. This proves the claim for $n+1$.

**Solution:**

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an $n$-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an $n$-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size $n$ graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph $G$ in which every vertex has degree at least 1. Remove an arbitrary vertex $v$, leaving an $n$-vertex graph $G'$ in which every vertex has degree... uh-oh! The reduced graph $G'$ might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —— and properly so, since the claim is false!

# 7 Graph Coloring

Prove that a graph with maximum degree at most $k$ is $(k+1)$-colorable. (Hint: consider inducting over the number of vertices.)

**Solution:**

The natural way to try to prove this theorem is to use induction on the graph's maximum degree, $k$. Unfortunately, this approach is extremely difficult because covering all possible types of graphs when maximum degree changes requires extreme caution. You might be envisioning a certain graph as you write your proof, but your argument will likely not generalize. In graphs, typical good choices for the induction parameter are $n$, the number of nodes, or $e$, the number of edges. We typically shy away from inducting on degree.

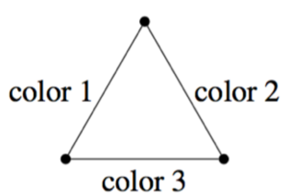
We use induction on the number of vertices in the graph, which we denote by $n$. Let $P(n)$ be the proposition that an $n$-vertex graph with maximum degree at most $k$ is $(k+1)$-colorable.

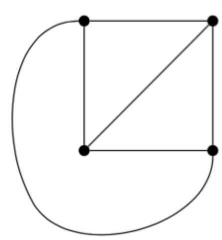
*Base Case $n = 1$:* A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

*Inductive Step:* Now assume that $P(n)$ is true, and let $G$ be an $(n+1)$-vertex graph with maximum degree at most $k$. Remove a vertex $v$ (and all edges incident to it), leaving an $n$-vertex subgraph,$H$. The maximum degree of $H$ is at most $k$, and so $H$ is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex $v$. We can assign $v$ a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most $k$ vertices adjacent to $v$ and so at least one of the $k+1$ colors is still available. Therefore, $G$ is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

# 8 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

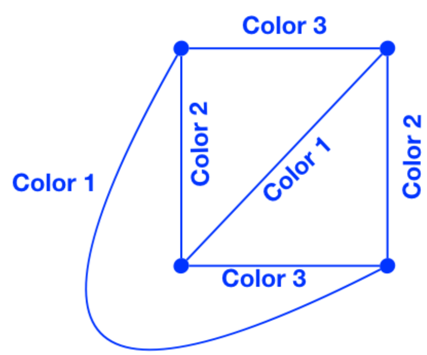
(c) Show that a tree can be edge colored with $d$ colors where $d$ is the maximum degree of any vertex.

**Solution:**

(a) Three color a triangle $u_1, u_2, u_3$ where $(u_1, u_2)$ is colored 1, $(u_2, u_3)$ is colored 2, and $(u_3, u_1)$ is colored 3. This is a valid 3 coloring as the edges are all colored differently.

Consider adding a fourth vertex $v$, the incident edges must be colored differently and each incident edge $(v, u_i)$ needs to be colored differently from the edges incident to $u_i$. That is, one can color $(v, u_1)$ with 2 as it is not incident to the edge colored 2 and that color is avalaible. Similarly one can color edge $(v, u_2)$ with color 3 and $(v, u_3)$ with color 1.

Another proof is simply provide a coloring which is below.




(b) We will use induction on the number of edges $n$ in the graph to prove the statement: If a graph $G$ has $n \geq 0$ edges and the maximum degree of any vertex is $d$, then $G$ can be colored with $2d-1$ colors.

*Base case ($n$ = 0).* If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

*Inductive hypothesis.* Suppose for $n = k \geq 0$, the statement holds.

*Inductive step.* Consider a graph $G$ with $n = k+1$ edges. Remove an edge of your choice, say $e$ from $G$. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge $e$. Thus, we can color edge $e$ without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices $n$ in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is $d$, then the tree can be colored with $d$ colors.

*Base case (n=1).* If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

*Inductive hypothesis.* Suppose the statement holds for $n = k \geq 1$.

*Inductive Step.* Remove any leaf $v$ of your choice from the tree. We can then color the remaining tree with $d$ colors by the inductive hypothesis. For any neighboring vertex $u$ of vertex $v$, the degree of $u$ is at most $d-1$ since we removed the edge $\{u,v\}$ along with the vertex $v$. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.